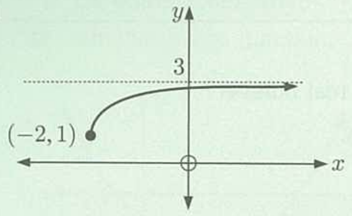
REVIEW SET 2C

1 For each of the following graphs, find the domain and range:

a



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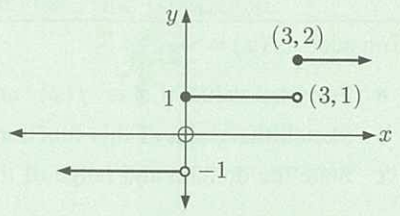


2 Given $f(x) = x^2 + 3$, find:

a $f(-3)$ **b** x such that $f(x) = 4$.

3 State the value(s) of x for which $f(x)$ is undefined:

a $f(x) = 10 + \dfrac{3}{2x-1}$ **b** $f(x) = \sqrt{x+7}$

4 Draw a sign diagram for:

a $f(x) = x(x+4)(3x+1)$ **b** $f(x) = \dfrac{-11}{(x+1)(x+8)}$

5 Given $h(x) = 7 - 3x$, find:

a $h(2x-1)$ in simplest form **b** x such that $h(2x-1) = -2$.

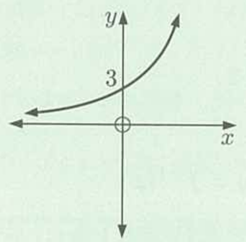
6 If $f(x) = 1 - 2x$ and $g(x) = \sqrt{x}$, find in simplest form:

a $(f \circ g)(x)$ **b** $(g \circ f)(x)$

7 Suppose $f(x) = ax^2 + bx + c$. Find a, b, and c if $f(0) = 5$, $f(-2) = 21$, and $f(3) = -4$.

8 Copy the following graphs and draw the graph of each inverse function on the same set of axes:

a



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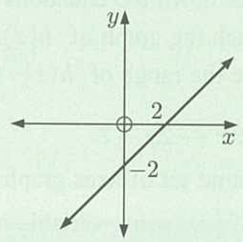


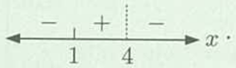
9 Find the inverse function $f^{-1}(x)$ for:

a $f(x) = 7 - 4x$ **b** $f(x) = \dfrac{3+2x}{5}$

10 Given $f : x \mapsto 5x - 2$ and $h : x \mapsto \dfrac{3x}{4}$, show that $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x)$.

11 Given $f(x) = 2x + 11$ and $g(x) = x^2$, find $(g \circ f^{-1})(3)$.

12 Sketch a function with domain $\{x \mid x \neq 4\}$, range $\{y \mid y \neq -1\}$, and sign diagram



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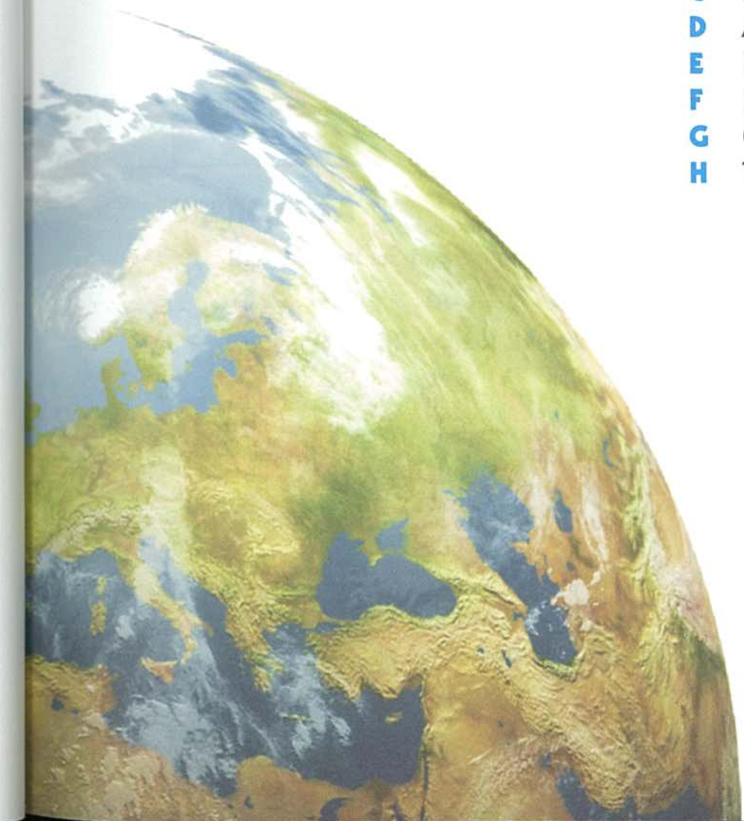
Chapter 3

Exponentials

Syllabus reference: 1.2, 2.2, 2.6, 2.7, 2.8

Contents:



OPENING PROBLEM

The interior of a freezer has temperature $-10°C$. When a packet of peas is placed in the freezer, its temperature after t minutes is given by $T(t) = -10 + 32 \times 2^{-0.2t}$ °C.

Things to think about:

a What was the temperature of the packet of peas:
- **i** when it was first placed in the freezer
- **ii** after 5 minutes
- **iii** after 10 minutes
- **iv** after 15 minutes?

b What does the graph of temperature over time look like?

c According to this model, will the temperature of the packet of peas ever reach $-10°C$? Explain your answer.

We often deal with numbers that are repeatedly multiplied together. Mathematicians use **exponents**, also called **powers**, or **indices**, to construct such expressions.

Exponents have many applications in the areas of finance, engineering, physics, electronics, biology, and computer science. Problems encountered in these areas may involve situations where quantities increase or decrease over time. Such problems are often examples of **exponential growth** or **decay**.

A EXPONENTS

Rather than writing $3 \times 3 \times 3 \times 3 \times 3$, we can write this product as 3^5.

If n is a positive integer, then a^n is the product of n factors of a.

$$a^n = \underbrace{a \times a \times a \times a \times \ ...... \ \times a}_{n \text{ factors}}$$

We say that a is the **base**, and n is the **exponent** or **index**.

3^5

base

power, index or exponent

NEGATIVE BASES

$(-1)^1 = -1$
$(-1)^2 = -1 \times -1 = 1$
$(-1)^3 = -1 \times -1 \times -1 = -1$
$(-1)^4 = -1 \times -1 \times -1 \times -1 = 1$

$(-2)^1 = -2$
$(-2)^2 = -2 \times -2 = 4$
$(-2)^3 = -2 \times -2 \times -2 = -8$
$(-2)^4 = -2 \times -2 \times -2 \times -2 = 16$

From the patterns above we can see that:

A **negative** base raised to an **odd** exponent is **negative**.
A **negative** base raised to an **even** exponent is **positive**.

CALCULATOR USE

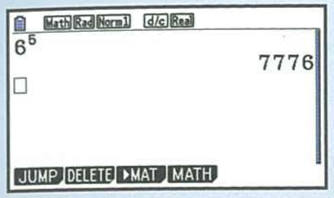
Although different calculators vary in the appearance of keys, they all perform operations of raising to powers in a similar manner. Click on the icon for instructions for calculating exponents.

GRAPHICS CALCULATOR INSTRUCTIONS

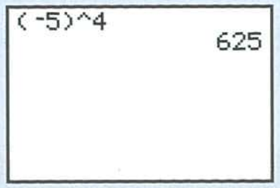
Example 1 Self Tutor

Find, using your calculator: **a** 6^5 **b** $(-5)^4$ **c** -7^4

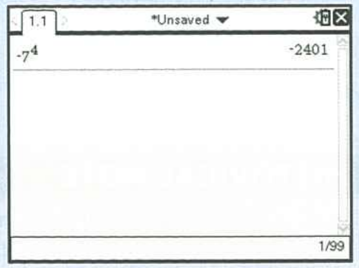
a Casio fx-CG20



b TI-84 Plus



c TI-nspire

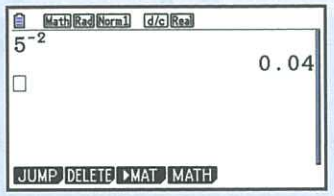


$6^5 = 7776$ $(-5)^4 = 625$ $-7^4 = -2401$

Example 2 Self Tutor

Find using your calculator: **a** 5^{-2} **b** $\frac{1}{5^2}$. Comment on your results.

a Casio fx-CG20



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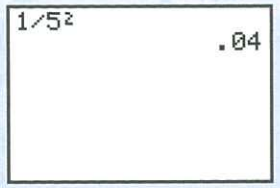


$5^{-2} = 0.04$ $\frac{1}{5^2} = 0.04$

The answers indicate that $5^{-2} = \frac{1}{5^2}$.

EXERCISE 3A

1 List the first six powers of:
a 2 **b** 3 **c** 4

2 Copy and complete the values of these common powers:
a $5^1 = ...,\ 5^2 = ...,\ 5^3 = ...,\ 5^4 = ...$
b $6^1 = ...,\ 6^2 = ...,\ 6^3 = ...,\ 6^4 = ...$
c $7^1 = ...,\ 7^2 = ...,\ 7^3 = ...,\ 7^4 = ...$

3 Simplify, then use a calculator to check your answer:
a $(-1)^5$ **b** $(-1)^6$ **c** $(-1)^{14}$ **d** $(-1)^{19}$ **e** $(-1)^8$ **f** -1^8
g $-(-1)^8$ **h** $(-2)^5$ **i** -2^5 **j** $-(-2)^6$ **k** $(-5)^4$ **l** $-(-5)^4$

4 Use your calculator to find the value of the following, recording the entire display:
a 4^7 **b** 7^4 **c** -5^5 **d** $(-5)^5$ **e** 8^6 **f** $(-8)^6$
g -8^6 **h** 2.13^9 **i** -2.13^9 **j** $(-2.13)^9$

5 Use your calculator to find the values of the following:

a 9^{-1} **b** $\frac{1}{9^1}$ **c** 6^{-2} **d** $\frac{1}{6^2}$ **e** 3^{-4} **f** $\frac{1}{3^4}$

g 17^0 **h** $(0.366)^0$

What do you notice?

6 Consider 3^1, 3^2, 3^3, 3^4, 3^5 .... Look for a pattern and hence find the last digit of 3^{101}.

7 What is the last digit of 7^{217}?

HISTORICAL NOTE

Nicomachus discovered an interesting number pattern involving cubes and sums of odd numbers. Nicomachus was born in Roman Syria (now Jerash, Jordan) around 100 AD. He wrote in Greek and was a Pythagorean.

$$1 = 1^3$$
$$3 + 5 = 8 = 2^3$$
$$7 + 9 + 11 = 27 = 3^3$$
$$\vdots$$

B LAWS OF EXPONENTS

The **exponent laws** for $m, n \in \mathbb{Z}$ are:

$a^m \times a^n = a^{m+n}$	To **multiply** numbers with the **same base**, keep the base and **add** the exponents.
$\dfrac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$	To **divide** numbers with the same base, keep the base and **subtract** the exponents.
$(a^m)^n = a^{m \times n}$	When **raising** a **power** to a **power**, keep the base and **multiply** the exponents.
$(ab)^n = a^n b^n$	The power of a product is the product of the powers.
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}, \quad b \neq 0$	The power of a quotient is the quotient of the powers.
$a^0 = 1, \quad a \neq 0$	Any non-zero number raised to the power of zero is 1.
$a^{-n} = \dfrac{1}{a^n}$ and $\dfrac{1}{a^{-n}} = a^n$	and in particular $a^{-1} = \dfrac{1}{a}, \quad a \neq 0$.

Example 3 — Self Tutor

Simplify using the exponent laws: **a** $3^5 \times 3^4$ **b** $\dfrac{5^3}{5^5}$ **c** $(m^4)^3$

a $3^5 \times 3^4$
$= 3^{5+4}$
$= 3^9$

b $\dfrac{5^3}{5^5}$
$= 5^{3-5}$
$= 5^{-2}$
$= \frac{1}{25}$

c $(m^4)^3$
$= m^{4 \times 3}$
$= m^{12}$

Example 4 — Self Tutor

Write as powers of 2:

a 16 **b** $\frac{1}{16}$ **c** 1 **d** 4×2^n **e** $\dfrac{2^m}{8}$

a 16
$= 2 \times 2 \times 2 \times 2$
$= 2^4$

b $\frac{1}{16}$
$= \dfrac{1}{2^4}$
$= 2^{-4}$

c 1
$= 2^0$

d 4×2^n
$= 2^2 \times 2^n$
$= 2^{2+n}$

e $\dfrac{2^m}{8}$
$= \dfrac{2^m}{2^3}$
$= 2^{m-3}$

EXERCISE 3B

1 Simplify using the laws of exponents:

a $5^4 \times 5^7$ **b** $d^2 \times d^6$ **c** $\dfrac{k^8}{k^3}$ **d** $\dfrac{7^5}{7^6}$ **e** $(x^2)^5$ **f** $(3^4)^4$

g $\dfrac{p^3}{p^7}$ **h** $n^3 \times n^9$ **i** $(5^t)^3$ **j** $7^x \times 7^2$ **k** $\dfrac{10^3}{10^q}$ **l** $(c^4)^m$

2 Write as powers of 2:

a 4 **b** $\frac{1}{4}$ **c** 8 **d** $\frac{1}{8}$ **e** 32 **f** $\frac{1}{32}$

g 2 **h** $\frac{1}{2}$ **i** 64 **j** $\frac{1}{64}$ **k** 128 **l** $\frac{1}{128}$

3 Write as powers of 3:

a 9 **b** $\frac{1}{9}$ **c** 27 **d** $\frac{1}{27}$ **e** 3 **f** $\frac{1}{3}$

g 81 **h** $\frac{1}{81}$ **i** 1 **j** 243 **k** $\frac{1}{243}$

4 Write as a single power of 2:

a 2×2^a **b** 4×2^b **c** 8×2^t **d** $(2^{x+1})^2$ **e** $(2^{1-n})^{-1}$

f $\dfrac{2^c}{4}$ **g** $\dfrac{2^m}{2^{-m}}$ **h** $\dfrac{4}{2^{1-n}}$ **i** $\dfrac{2^{x+1}}{2^x}$ **j** $\dfrac{4^x}{2^{1-x}}$

5 Write as a single power of 3:

a 9×3^p **b** 27^a **c** 3×9^n **d** 27×3^d **e** 9×27^t

f $\dfrac{3^y}{3}$ **g** $\dfrac{3}{3^y}$ **h** $\dfrac{9}{27^t}$ **i** $\dfrac{9^a}{3^{1-a}}$ **j** $\dfrac{9^{n+1}}{3^{2n-1}}$

Example 5 — Self Tutor

Write in simplest form, without brackets:

a $(-3a^2)^4$ **b** $\left(-\dfrac{2a^2}{b}\right)^3$

a $(-3a^2)^4$
$= (-3)^4 \times (a^2)^4$
$= 81 \times a^{2\times4}$
$= 81a^8$

b $\left(-\dfrac{2a^2}{b}\right)^3$
$= \dfrac{(-2)^3 \times (a^2)^3}{b^3}$
$= \dfrac{-8a^6}{b^3}$

6 Write without brackets:

a $(2a)^2$ **b** $(3b)^3$ **c** $(ab)^4$ **d** $(pq)^3$ **e** $\left(\frac{m}{n}\right)^2$

f $\left(\frac{a}{3}\right)^3$ **g** $\left(\frac{b}{c}\right)^4$ **h** $\left(\frac{2a}{b}\right)^0$ **i** $\left(\frac{m}{3n}\right)^4$ **j** $\left(\frac{xy}{2}\right)^3$

7 Write the following in simplest form, without brackets:

a $(-2a)^2$ **b** $(-6b^2)^2$ **c** $(-2a)^3$ **d** $(-3m^2n^2)^3$

e $(-2ab^4)^4$ **f** $\left(\frac{-2a^2}{b^2}\right)^3$ **g** $\left(\frac{-4a^3}{b}\right)^2$ **h** $\left(\frac{-3p^2}{q^3}\right)^2$

Example 6 Self Tutor

Write without negative exponents: $\dfrac{a^{-3}b^2}{c^{-1}}$

$a^{-3} = \dfrac{1}{a^3}$ and $\dfrac{1}{c^{-1}} = c^1$

$\therefore \dfrac{a^{-3}b^2}{c^{-1}} = \dfrac{b^2c}{a^3}$

8 Write without negative exponents:

a ab^{-2} **b** $(ab)^{-2}$ **c** $(2ab^{-1})^2$ **d** $(3a^{-2}b)^2$ **e** $\dfrac{a^2b^{-1}}{c^2}$

f $\dfrac{a^2b^{-1}}{c^{-2}}$ **g** $\dfrac{1}{a^{-3}}$ **h** $\dfrac{a^{-2}}{b^{-3}}$ **i** $\dfrac{2a^{-1}}{d^2}$ **j** $\dfrac{12a}{m^{-3}}$

Example 7 Self Tutor

Write $\dfrac{1}{2^{1-n}}$ in non-fractional form.

$$\frac{1}{2^{1-n}} = 2^{-(1-n)}$$
$$= 2^{-1+n}$$
$$= 2^{n-1}$$

9 Write in non-fractional form:

a $\dfrac{1}{a^n}$ **b** $\dfrac{1}{b^{-n}}$ **c** $\dfrac{1}{3^{2-n}}$ **d** $\dfrac{a^n}{b^{-m}}$ **e** $\dfrac{a^{-n}}{a^{2+n}}$

10 Simplify, giving your answers in simplest rational form:

a $\left(\frac{5}{3}\right)^0$ **b** $\left(\frac{7}{4}\right)^{-1}$ **c** $\left(\frac{1}{6}\right)^{-1}$ **d** $\dfrac{3^3}{3^0}$

e $\left(\frac{4}{3}\right)^{-2}$ **f** $2^1 + 2^{-1}$ **g** $\left(1\frac{2}{3}\right)^{-3}$ **h** $5^2 + 5^1 + 5^{-1}$

11 Write as powers of 2, 3 and/or 5:

a $\frac{1}{9}$ **b** $\frac{1}{16}$ **c** $\frac{1}{125}$ **d** $\frac{3}{5}$

e $\frac{4}{27}$ **f** $\dfrac{2^c}{8 \times 9}$ **g** $\dfrac{9^k}{10}$ **h** $\dfrac{6^p}{75}$

12 Read about Nicomachus' pattern on page **84** and find the series of odd numbers for:

a 5^3 **b** 7^3 **c** 12^3

C RATIONAL EXPONENTS

The exponent laws used previously can also be applied to **rational exponents**, or exponents which are written as a fraction.

For $a > 0$, notice that $a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a$ {exponent laws}
and $\sqrt{a} \times \sqrt{a} = a$ also.

So, $\boldsymbol{a^{\frac{1}{2}} = \sqrt{a}}$ {by direct comparison}

Likewise $a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^1 = a$
and $\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a$

suggests $\boldsymbol{a^{\frac{1}{3}} = \sqrt[3]{a}}$

In general, $\boldsymbol{a^{\frac{1}{n}} = \sqrt[n]{a}}$ where $\sqrt[n]{a}$ reads 'the nth root of a', for $n \in \mathbb{Z}^+$.

We can now determine that $\sqrt[n]{a^m}$
$= (a^m)^{\frac{1}{n}}$
$= a^{\frac{m}{n}}$ $\quad\therefore\quad$ $\boldsymbol{a^{\frac{m}{n}} = \sqrt[n]{a^m}}$ for $a > 0$, $n \in \mathbb{Z}^+$, $m \in \mathbb{Z}$

Example 8 Self Tutor

Write as a single power of 2: **a** $\sqrt[3]{2}$ **b** $\dfrac{1}{\sqrt{2}}$ **c** $\sqrt[5]{4}$

a $\sqrt[3]{2}$
$= 2^{\frac{1}{3}}$

b $\dfrac{1}{\sqrt{2}}$
$= \dfrac{1}{2^{\frac{1}{2}}}$
$= 2^{-\frac{1}{2}}$

c $\sqrt[5]{4}$
$= (2^2)^{\frac{1}{5}}$
$= 2^{2 \times \frac{1}{5}}$
$= 2^{\frac{2}{5}}$

EXERCISE 3C

1 Write as a single power of 2:

a $\sqrt[5]{2}$ **b** $\dfrac{1}{\sqrt[5]{2}}$ **c** $2\sqrt{2}$ **d** $4\sqrt{2}$ **e** $\dfrac{1}{\sqrt[3]{2}}$

f $2 \times \sqrt[3]{2}$ **g** $\dfrac{4}{\sqrt{2}}$ **h** $(\sqrt{2})^3$ **i** $\dfrac{1}{\sqrt[3]{16}}$ **j** $\dfrac{1}{\sqrt{8}}$

2 Write as a single power of 3:

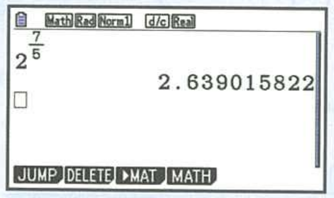
a $\sqrt[3]{3}$ **b** $\dfrac{1}{\sqrt[3]{3}}$ **c** $\sqrt[4]{3}$ **d** $3\sqrt{3}$ **e** $\dfrac{1}{9\sqrt{3}}$

Example 9 — Self Tutor

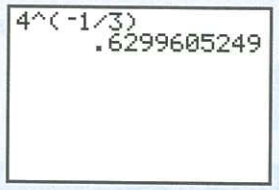
Use your calculator to evaluate:

a $2^{\frac{7}{5}}$ **b** $\dfrac{1}{\sqrt[3]{4}}$

a Casio fx-CG20



b TI-84 Plus



$2^{\frac{7}{5}} \approx 2.639\,016$

$\dfrac{1}{\sqrt[3]{4}} = 4^{-\frac{1}{3}} \approx 0.629\,961$

Example 10 — Self Tutor

Without using a calculator, write in simplest rational form:

a $8^{\frac{4}{3}}$ **b** $27^{-\frac{2}{3}}$

a
$$\begin{aligned} & 8^{\frac{4}{3}} \\ &= (2^3)^{\frac{4}{3}} \\ &= 2^{3\times\frac{4}{3}} \qquad \{(a^m)^n = a^{mn}\} \\ &= 2^4 \\ &= 16 \end{aligned}$$

b
$$\begin{aligned} & 27^{-\frac{2}{3}} \\ &= (3^3)^{-\frac{2}{3}} \\ &= 3^{3\times-\frac{2}{3}} \\ &= 3^{-2} \\ &= \tfrac{1}{9} \end{aligned}$$

3 Write the following in the form a^x where a is a prime number and x is rational:

a $\sqrt[3]{7}$ **b** $\sqrt[4]{27}$ **c** $\sqrt[5]{16}$ **d** $\sqrt[3]{32}$ **e** $\sqrt[7]{49}$

f $\dfrac{1}{\sqrt[3]{7}}$ **g** $\dfrac{1}{\sqrt[4]{27}}$ **h** $\dfrac{1}{\sqrt[5]{16}}$ **i** $\dfrac{1}{\sqrt[3]{32}}$ **j** $\dfrac{1}{\sqrt[7]{49}}$

4 Use your calculator to find:

a $3^{\frac{3}{4}}$ **b** $2^{\frac{7}{8}}$ **c** $2^{-\frac{1}{3}}$ **d** $4^{-\frac{3}{5}}$ **e** $\sqrt[4]{8}$

f $\sqrt[5]{27}$ **g** $\dfrac{1}{\sqrt[3]{7}}$

5 Without using a calculator, write in simplest rational form:

a $4^{\frac{3}{2}}$ **b** $8^{\frac{5}{3}}$ **c** $16^{\frac{3}{4}}$ **d** $25^{\frac{3}{2}}$ **e** $32^{\frac{2}{5}}$

f $4^{-\frac{1}{2}}$ **g** $9^{-\frac{3}{2}}$ **h** $8^{-\frac{4}{3}}$ **i** $27^{-\frac{4}{3}}$ **j** $125^{-\frac{2}{3}}$

THEORY OF KNOWLEDGE

A **rational number** is a number which can be written in the form $\frac{p}{q}$ where p and q are integers, $q \neq 0$. It has been proven that a rational number has a decimal expansion which either terminates or recurs.

If we begin to write the decimal expansion of $\sqrt{2}$, there is no indication that it will terminate or recur, and we might therefore suspect that $\sqrt{2}$ is irrational.

1.414 213 562 373 095 048 801 688 724 209 698 078 569 671 875 376 948 073 ....

However, we cannot *prove* that $\sqrt{2}$ is irrational by writing out its decimal expansion, as we would have to write an infinite number of decimal places. We might therefore *believe* that $\sqrt{2}$ is irrational, but it may also seem impossible to *prove* it.

1 If something has not yet been proven, does that make it untrue?

2 Is the state of an idea being true or false dependent on our ability to prove it?

In fact, we can quite easily prove that $\sqrt{2}$ is irrational by using a method called **proof by contradiction**. In this method we suppose that the opposite is true of what we want to show is true, and follow a series of logical steps until we arrive at a contradiction. The contradiction confirms that our original supposition must be false.

Proof: Suppose $\sqrt{2}$ is rational, so $\sqrt{2} = \frac{p}{q}$ for some (positive) integers p and q, $q \neq 0$.

We assume this fraction has been written in lowest terms, so p and q have no common factors.

Squaring both sides, $2 = \dfrac{p^2}{q^2}$

$\therefore\ p^2 = 2q^2$ .... (1)

$\therefore\ p^2$ is even, and so p must be even.

Thus $p = 2k$ for some $k \in \mathbb{Z}^+$.

Substituting into (1), $4k^2 = 2q^2$

$\therefore\ q^2 = 2k^2$

$\therefore\ q^2$ is even, and so q must be even.

Here we have a contradiction, as p and q have no common factors.

Thus our original supposition is false, and $\sqrt{2}$ is irrational.

3 Is proof by contradiction unique to mathematics, or do we use it elsewhere?

D ALGEBRAIC EXPANSION AND FACTORISATION

EXPANSION

We can use the usual expansion laws to simplify expressions containing exponents:

$$\begin{aligned} a(b+c) &= ab+ac \\ (a+b)(c+d) &= ac+ad+bc+bd \\ (a+b)(a-b) &= a^2-b^2 \\ (a+b)^2 &= a^2+2ab+b^2 \\ (a-b)^2 &= a^2-2ab+b^2 \end{aligned}$$

Example 11 Self Tutor

Expand and simplify: $x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}})$

$$\begin{aligned} & x^{-\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}-3x^{-\frac{1}{2}}) \\ &= x^{-\frac{1}{2}}\times x^{\frac{3}{2}}+x^{-\frac{1}{2}}\times 2x^{\frac{1}{2}}-x^{-\frac{1}{2}}\times 3x^{-\frac{1}{2}} && \{\text{each term is } \times \text{ by } x^{-\frac{1}{2}}\} \\ &= x^1+2x^0-3x^{-1} && \{\text{adding exponents}\} \\ &= x+2-\frac{3}{x} \end{aligned}$$

Example 12 Self Tutor

Expand and simplify: **a** $(2^x+3)(2^x+1)$ **b** $(7^x+7^{-x})^2$

a
$$\begin{aligned} & (2^x+3)(2^x+1) \\ &= 2^x\times 2^x+2^x+3\times 2^x+3 \\ &= 2^{2x}+4\times 2^x+3 \\ &= 4^x+2^{2+x}+3 \end{aligned}$$

b
$$\begin{aligned} & (7^x+7^{-x})^2 \\ &= (7^x)^2+2\times 7^x\times 7^{-x}+(7^{-x})^2 \\ &= 7^{2x}+2\times 7^0+7^{-2x} \\ &= 7^{2x}+2+7^{-2x} \end{aligned}$$

EXERCISE 3D.1

1. Expand and simplify:
   - **a** $x^2(x^3+2x^2+1)$
   - **b** $2^x(2^x+1)$
   - **c** $x^{\frac{1}{2}}(x^{\frac{1}{2}}+x^{-\frac{1}{2}})$
   - **d** $7^x(7^x+2)$
   - **e** $3^x(2-3^{-x})$
   - **f** $x^{\frac{1}{2}}(x^{\frac{3}{2}}+2x^{\frac{1}{2}}+3x^{-\frac{1}{2}})$
   - **g** $2^{-x}(2^x+5)$
   - **h** $5^{-x}(5^{2x}+5^x)$
   - **i** $x^{-\frac{1}{2}}(x^2+x+x^{\frac{1}{2}})$
2. Expand and simplify:
   - **a** $(2^x-1)(2^x+3)$
   - **b** $(3^x+2)(3^x+5)$
   - **c** $(5^x-2)(5^x-4)$
   - **d** $(2^x+3)^2$
   - **e** $(3^x-1)^2$
   - **f** $(4^x+7)^2$
   - **g** $(x^{\frac{1}{2}}+2)(x^{\frac{1}{2}}-2)$
   - **h** $(2^x+3)(2^x-3)$
   - **i** $(x^{\frac{1}{2}}+x^{-\frac{1}{2}})(x^{\frac{1}{2}}-x^{-\frac{1}{2}})$
   - **j** $(x+\frac{2}{x})^2$
   - **k** $(7^x-7^{-x})^2$
   - **l** $(5-2^{-x})^2$

FACTORISATION AND SIMPLIFICATION

Example 13 Self Tutor

Factorise: **a** $2^{n+3}+2^n$ **b** $2^{n+3}+8$ **c** $2^{3n}+2^{2n}$

a
$$\begin{aligned} & 2^{n+3}+2^n \\ &= 2^n2^3+2^n \\ &= 2^n(2^3+1) \\ &= 2^n\times 9 \end{aligned}$$

b
$$\begin{aligned} & 2^{n+3}+8 \\ &= 2^n2^3+8 \\ &= 8(2^n)+8 \\ &= 8(2^n+1) \end{aligned}$$

c
$$\begin{aligned} & 2^{3n}+2^{2n} \\ &= 2^{2n}2^n+2^{2n} \\ &= 2^{2n}(2^n+1) \end{aligned}$$

Example 14 Self Tutor

Factorise: **a** 4^x-9 **b** $9^x+4(3^x)+4$

a
$$\begin{aligned} & 4^x-9 \\ &= (2^x)^2-3^2 && \{\text{compare } a^2-b^2=(a+b)(a-b)\} \\ &= (2^x+3)(2^x-3) \end{aligned}$$

b
$$\begin{aligned} & 9^x+4(3^x)+4 \\ &= (3^x)^2+4(3^x)+4 && \{\text{compare } a^2+4a+4\} \\ &= (3^x+2)^2 && \{\text{as } a^2+4a+4=(a+2)^2\} \end{aligned}$$

EXERCISE 3D.2

1. Factorise:
   - **a** $5^{2x}+5^x$
   - **b** $3^{n+2}+3^n$
   - **c** 7^n+7^{3n}
   - **d** $5^{n+1}-5$
   - **e** $6^{n+2}-6$
   - **f** $4^{n+2}-16$
2. Factorise:
   - **a** 9^x-4
   - **b** 4^x-25
   - **c** $16-9^x$
   - **d** $25-4^x$
   - **e** 9^x-4^x
   - **f** $4^x+6(2^x)+9$
   - **g** $9^x+10(3^x)+25$
   - **h** $4^x-14(2^x)+49$
   - **i** $25^x-4(5^x)+4$
3. Factorise:
   - **a** $4^x+9(2^x)+18$
   - **b** 4^x-2^x-20
   - **c** $9^x+9(3^x)+14$
   - **d** $9^x+4(3^x)-5$
   - **e** 25^x+5^x-2
   - **f** $49^x-7^{x+1}+12$

Example 15 Self Tutor

Simplify: **a** $\frac{6^n}{3^n}$ **b** $\frac{4^n}{6^n}$

a
$$\frac{6^n}{3^n} = \frac{2^n\cancel{3^n}}{\cancel{3^n}_1} = 2^n \qquad \text{or} \qquad \frac{6^n}{3^n} = \left(\frac{6}{3}\right)^n = 2^n$$

b
$$\frac{4^n}{6^n} = \frac{\cancel{2^n}2^n}{\cancel{2^n}3^n} = \frac{2^n}{3^n} \qquad \text{or} \qquad \frac{4^n}{6^n} = \left(\frac{4}{6}\right)^n = \left(\frac{2}{3}\right)^n$$

Example 16 — Self Tutor

Simplify: **a** $\dfrac{3^n+6^n}{3^n}$ **b** $\dfrac{2^{m+2}-2^m}{2^m}$ **c** $\dfrac{2^{m+3}+2^m}{9}$

a
$$\frac{3^n+6^n}{3^n} = \frac{3^n+2^n3^n}{3^n} = \frac{\cancel{3^n}(1+2^n)}{\cancel{3^n}_1} = 1+2^n$$

b
$$\frac{2^{m+2}-2^m}{2^m} = \frac{2^m2^2-2^m}{2^m} = \frac{\cancel{2^m}(4-1)}{\cancel{2^m}_1} = 3$$

c
$$\frac{2^{m+3}+2^m}{9} = \frac{2^m2^3+2^m}{9} = \frac{2^m\cancel{(8+1)}}{\cancel{9}_1} = 2^m$$

4 Simplify:

a $\dfrac{12^n}{6^n}$ **b** $\dfrac{20^a}{2^a}$ **c** $\dfrac{6^b}{2^b}$ **d** $\dfrac{4^n}{20^n}$

e $\dfrac{35^x}{7^x}$ **f** $\dfrac{6^a}{8^a}$ **g** $\dfrac{5^{n+1}}{5^n}$ **h** $\dfrac{5^{n+1}}{5}$

5 Simplify:

a $\dfrac{6^m+2^m}{2^m}$ **b** $\dfrac{2^n+12^n}{2^n}$ **c** $\dfrac{8^n+4^n}{2^n}$

d $\dfrac{12^x-3^x}{3^x}$ **e** $\dfrac{6^n+12^n}{1+2^n}$ **f** $\dfrac{5^{n+1}-5^n}{4}$

g $\dfrac{5^{n+1}-5^n}{5^n}$ **h** $\dfrac{4^n-2^n}{2^n}$ **i** $\dfrac{2^n-2^{n-1}}{2^n}$

6 Simplify:

a $2^n(n+1)+2^n(n-1)$ **b** $3^n\left(\dfrac{n-1}{6}\right)-3^n\left(\dfrac{n+1}{6}\right)$

E EXPONENTIAL EQUATIONS

An **exponential equation** is an equation in which the unknown occurs as part of the index or exponent.

For example: $2^x=8$ and $30\times 3^x=7$ are both exponential equations.

There are a number of methods we can use to solve exponential equations. These include graphing, using technology, and by using **logarithms**, which we will study in **Chapter 4**. However, in some cases we can solve algebraically by the following observation:

If $2^x=8$ then $2^x=2^3$. Thus $x=3$, and this is the only solution.

If the base numbers are the same, we can **equate exponents**.
If $a^x=a^k$ then $x=k$.

Example 17 — Self Tutor

Solve for x: **a** $2^x=16$ **b** $3^{x+2}=\frac{1}{27}$

a
$$\begin{aligned} 2^x&=16\\ \therefore\ 2^x&=2^4\\ \therefore\ x&=4\end{aligned}$$

b
$$\begin{aligned} 3^{x+2}&=\tfrac{1}{27}\\ \therefore\ 3^{x+2}&=3^{-3}\\ \therefore\ x+2&=-3\\ \therefore\ x&=-5\end{aligned}$$

Once we have the same base we then equate the exponents.

Example 18 — Self Tutor

Solve for x: **a** $4^x=8$ **b** $9^{x-2}=\frac{1}{3}$

a
$$\begin{aligned} 4^x&=8\\ \therefore\ (2^2)^x&=2^3\\ \therefore\ 2^{2x}&=2^3\\ \therefore\ 2x&=3\\ \therefore\ x&=\tfrac{3}{2}\end{aligned}$$

b
$$\begin{aligned} 9^{x-2}&=\tfrac{1}{3}\\ \therefore\ (3^2)^{x-2}&=3^{-1}\\ \therefore\ 3^{2(x-2)}&=3^{-1}\\ \therefore\ 2(x-2)&=-1\\ \therefore\ 2x-4&=-1\\ \therefore\ 2x&=3\\ \therefore\ x&=\tfrac{3}{2}\end{aligned}$$

EXERCISE 3E

1 Solve for x:

a $2^x=8$ **b** $5^x=25$ **c** $3^x=81$ **d** $7^x=1$

e $3^x=\frac{1}{3}$ **f** $2^x=\sqrt{2}$ **g** $5^x=\frac{1}{125}$ **h** $4^{x+1}=64$

i $2^{x-2}=\frac{1}{32}$ **j** $3^{x+1}=\frac{1}{27}$ **k** $7^{x+1}=343$ **l** $5^{1-2x}=\frac{1}{5}$

2 Solve for x:

a $8^x=32$ **b** $4^x=\frac{1}{8}$ **c** $9^x=\frac{1}{27}$ **d** $25^x=\frac{1}{5}$

e $27^x=\frac{1}{9}$ **f** $16^x=\frac{1}{32}$ **g** $4^{x+2}=128$ **h** $25^{1-x}=\frac{1}{125}$

i $4^{4x-1}=\frac{1}{2}$ **j** $9^{x-3}=27$ **k** $(\frac{1}{2})^{x+1}=8$ **l** $(\frac{1}{3})^{x+2}=9$

m $81^x=27^{-x}$ **n** $(\frac{1}{4})^{1-x}=32$ **o** $(\frac{1}{7})^x=49$ **p** $(\frac{1}{3})^{x+1}=243$

3 Solve for x, if possible:

a $4^{2x+1}=8^{1-x}$ **b** $9^{2-x}=(\frac{1}{3})^{2x+1}$ **c** $2^x\times 8^{1-x}=\frac{1}{4}$

4 Solve for x:

a $3\times 2^x=24$ **b** $7\times 2^x=56$ **c** $3\times 2^{x+1}=24$

d $12\times 3^{-x}=\frac{4}{3}$ **e** $4\times(\frac{1}{3})^x=36$ **f** $5\times(\frac{1}{2})^x=20$

Example 19 Self Tutor

Solve for x: $4^x + 2^x - 20 = 0$

$$\begin{aligned}
4^x + 2^x - 20 &= 0 \\
\therefore\ (2^x)^2 + 2^x - 20 &= 0 \qquad \{\text{compare } a^2 + a - 20 = 0\} \\
\therefore\ (2^x - 4)(2^x + 5) &= 0 \qquad \{a^2 + a - 20 = (a-4)(a+5)\} \\
\therefore\ 2^x = 4 &\text{ or } 2^x = -5 \\
\therefore\ 2^x &= 2^2 \qquad \{2^x \text{ cannot be negative}\} \\
\therefore\ x &= 2
\end{aligned}$$

5 Solve for x:

- **a** $4^x - 6(2^x) + 8 = 0$
- **b** $4^x - 2^x - 2 = 0$
- **c** $9^x - 12(3^x) + 27 = 0$
- **d** $9^x = 3^x + 6$
- **e** $25^x - 23(5^x) - 50 = 0$
- **f** $49^x + 1 = 2(7^x)$

Check your answers using technology. You can get instructions for doing this by clicking on the icon.

GRAPHICS CALCULATOR INSTRUCTIONS

F EXPONENTIAL FUNCTIONS

We have already seen how to evaluate b^n when $n \in \mathbb{Q}$, or in other words when n is a rational number.

But what about b^n when $n \in \mathbb{R}$, so n is real but not necessarily rational?

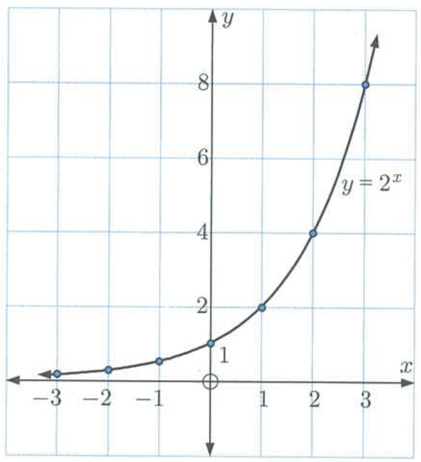
To answer this question, we can look at graphs of exponential functions.

The most simple general **exponential function** has the form $y = b^x$ where $b > 0$, $b \neq 1$.

For example, $y = 2^x$ is an exponential function.

We construct a table of values from which we graph the function:

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8



When $x = -10$, $y = 2^{-10} \approx 0.001$.
When $x = -50$, $y = 2^{-50} \approx 8.88 \times 10^{-16}$.

As x becomes large and negative, the graph of $y = 2^x$ approaches the x-axis from above but never touches it, since 2^x becomes very small but never zero.

So, as $x \to -\infty$, $y \to 0^+$.

We say that $y = 2^x$ is '**asymptotic** to the x-axis' or '$y = 0$ is a **horizontal asymptote**'.

We now have a well-defined meaning for b^n where $b, n \in \mathbb{R}$ because simple exponential functions have smooth increasing or decreasing graphs.

INVESTIGATION 1 GRAPHS OF EXPONENTIAL FUNCTIONS

In this investigation we examine the graphs of various families of exponential functions.

Click on the icon to run the **dynamic graphing package**, or else you could use your **graphics calculator**.

DYNAMIC GRAPHING PACKAGE

What to do:

1 Explore the family of curves of the form $y = b^x$ where $b > 0$.
For example, consider $y = 2^x$, $y = 3^x$, $y = 10^x$, and $y = (1.3)^x$.

- **a** What effect does changing b have on the shape of the graph?
- **b** What is the y-intercept of each graph?
- **c** What is the horizontal asymptote of each graph?

2 Explore the family of curves of the form $y = 2^x + d$ where d is a constant.
For example, consider $y = 2^x$, $y = 2^x + 1$, and $y = 2^x - 2$.

- **a** What effect does changing d have on the position of the graph?
- **b** What effect does changing d have on the shape of the graph?
- **c** What is the horizontal asymptote of each graph?
- **d** What is the horizontal asymptote of $y = 2^x + d$?
- **e** To graph $y = 2^x + d$ from $y = 2^x$ what transformation is used?

3 Explore the family of curves of the form $y = 2^{x-c}$.
For example, consider $y = 2^x$, $y = 2^{x-1}$, $y = 2^{x+2}$, and $y = 2^{x-3}$.

- **a** What effect does changing c have on the position of the graph?
- **b** What effect does changing c have on the shape of the graph?
- **c** What is the horizontal asymptote of each graph?
- **d** To graph $y = 2^{x-c}$ from $y = 2^x$ what transformation is used?

4 Explore the relationship between $y = b^x$ and $y = b^{-x}$ where $b > 0$.
For example, consider $y = 2^x$ and $y = 2^{-x}$.

- **a** What is the y-intercept of each graph?
- **b** What is the horizontal asymptote of each graph?
- **c** What transformation moves $y = 2^x$ to $y = 2^{-x}$?

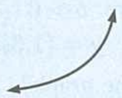
5 Explore the family of curves of the form $y = a \times 2^x$ where a is a constant.

- **a** Consider functions where $a > 0$, such as $y = 2^x$, $y = 3 \times 2^x$, and $y = \frac{1}{2} \times 2^x$.
Comment on the effect on the graph.
- **b** Consider functions where $a < 0$, such as $y = -2^x$, $y = -3 \times 2^x$, and $y = -\frac{1}{2} \times 2^x$.
Comment on the effect on the graph.
- **c** What is the horizontal asymptote of each graph? Explain your answer.

From your investigation you should have discovered that:

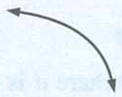
For the general exponential function $y = a \times b^{x-c} + d$ where $b > 0$, $b \neq 1$, $a \neq 0$:

- b controls how steeply the graph increases or decreases
- c controls horizontal translation
- d controls vertical translation
- the equation of the horizontal asymptote is $y = d$
- if $a > 0$, $b > 1$ the function is increasing.

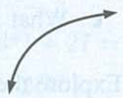


- if $a > 0$, $0 < b < 1$ the function is decreasing.

- if $a < 0$, $b > 1$ the function is decreasing.

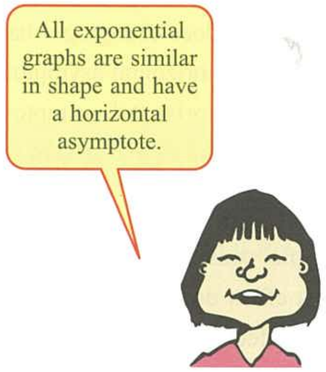


- if $a < 0$, $0 < b < 1$ the function is increasing.



We can sketch reasonably accurate graphs of exponential functions using:

- the horizontal asymptote
- the y-intercept
- two other points, for example, when $x = 2$, $x = -2$



Example 20 **Self Tutor**

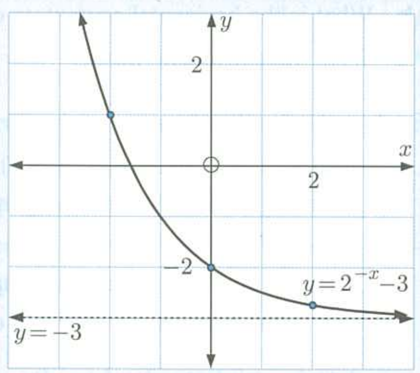
Sketch the graph of $y = 2^{-x} - 3$.
Hence state the domain and range of $f(x) = 2^{-x} - 3$.

For $y = 2^{-x} - 3$,
the horizontal asymptote is $y = -3$.

When $x = 0$, $y = 2^0 - 3$
$= 1 - 3$
$= -2$
$\therefore$ the y-intercept is -2.

When $x = 2$, $y = 2^{-2} - 3$
$= \frac{1}{4} - 3$
$= -2\frac{3}{4}$

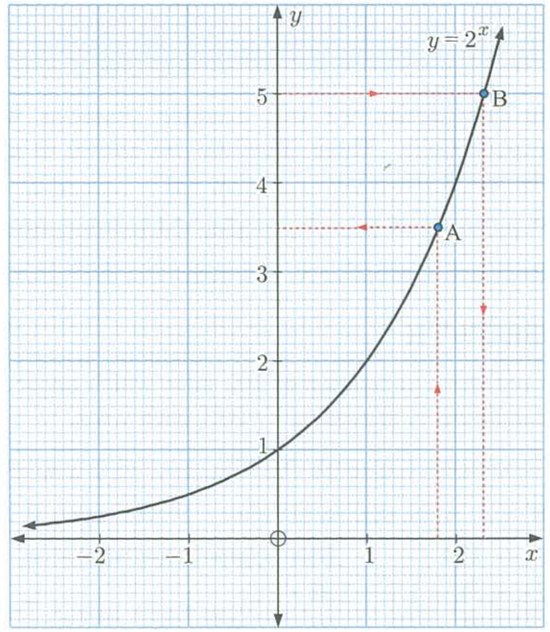
When $x = -2$, $y = 2^2 - 3 = 1$



The domain is $\{x \mid x \in \mathbb{R}\}$. The range is $\{y \mid y > -3\}$.

Consider the graph of $y = 2^x$ alongside. We can use the graph to estimate:

- the value of 2^x for a given value of x, for example $2^{1.8} \approx 3.5$ {point A}
- the solutions of the exponential equation $2^x = b$, for example if $2^x = 5$ then $x \approx 2.3$ {point B}.



EXERCISE 3F

1 Use the graph above to estimate the value of:

a $2^{\frac{1}{2}}$ or $\sqrt{2}$ **b** $2^{0.8}$ **c** $2^{1.5}$ **d** $2^{-\sqrt{2}}$

2 Use the graph above to estimate the solution to:

a $2^x = 3$ **b** $2^x = 0.6$

3 Use the graph of $y = 2^x$ to explain why $2^x = 0$ has no solutions.

4 Draw freehand sketches of the following pairs of graphs using your observations from the previous investigation:

GRAPHING PACKAGE

a $y = 2^x$ and $y = 2^x - 2$ **b** $y = 2^x$ and $y = 2^{-x}$
c $y = 2^x$ and $y = 2^{x-2}$ **d** $y = 2^x$ and $y = 2(2^x)$

5 Draw freehand sketches of the following pairs of graphs:

a $y = 3^x$ and $y = 3^{-x}$ **b** $y = 3^x$ and $y = 3^x + 1$
c $y = 3^x$ and $y = -3^x$ **d** $y = 3^x$ and $y = 3^{x-1}$

6 For each of the functions below:

i sketch the graph of the function
ii state the domain and range
iii use your calculator to find the value of y when $x = \sqrt{2}$
iv discuss the behaviour of y as $x \to \pm\infty$
v determine the horizontal asymptotes.

a $y = 2^x + 1$ **b** $y = 2 - 2^x$ **c** $y = 2^{-x} + 3$ **d** $y = 3 - 2^{-x}$

G GROWTH AND DECAY

In this section we will examine situations where quantities are either increasing or decreasing exponentially. These situations are known as **growth** and **decay** modelling, and occur frequently in the world around us.

Populations of animals, people, and bacteria usually *grow* in an exponential way.

Radioactive substances, and items that depreciate in value, usually *decay* exponentially.

GROWTH

Consider a population of 100 mice which under favourable conditions is increasing by 20% each week.

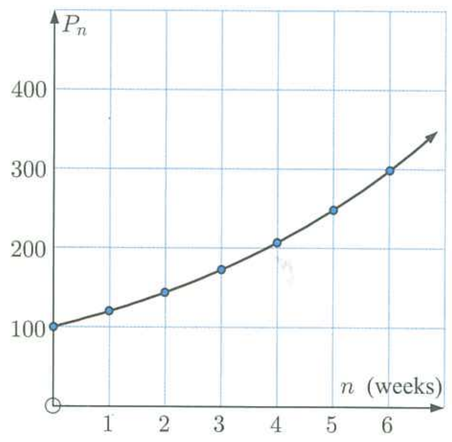
To increase a quantity by 20%, we multiply it by 1.2.

If P_n is the population after n weeks, then:

$P_0 = 100$ {the *original* population}
$P_1 = P_0 \times 1.2 = 100 \times 1.2$
$P_2 = P_1 \times 1.2 = 100 \times (1.2)^2$
$P_3 = P_2 \times 1.2 = 100 \times (1.2)^3$, and so on.

From this pattern we see that $P_n = 100 \times (1.2)^n$.

So, the graph of the population is a smooth curve given by the exponential function $P_n = 100 \times (1.2)^n$.



Example 21 Self Tutor

An entomologist monitoring a grasshopper plague notices that the area affected by the grasshoppers is given by $A_n = 1000 \times 2^{0.2n}$ hectares, where n is the number of weeks after the initial observation.

a Find the original affected area.

b Find the affected area after: **i** 5 weeks **ii** 10 weeks **iii** 12 weeks.

c Draw the graph of A_n against n.

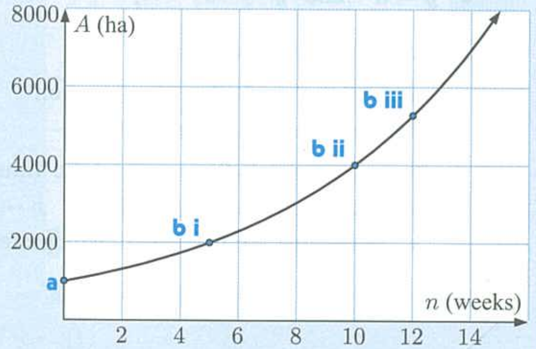
a $A_0 = 1000 \times 2^0$
$= 1000 \times 1$
$= 1000$ $\therefore$ the original affected area was 1000 ha.

b **i** $A_5 = 1000 \times 2^1$
$= 2000$
The affected area is 2000 ha.

ii $A_{10} = 1000 \times 2^2$
$= 4000$
The affected area is 4000 ha.

iii $A_{12} = 1000 \times 2^{0.2 \times 12}$
$= 1000 \times 2^{2.4}$
≈ 5280
The area affected is about 5280 ha.

c



EXERCISE 3G.1

GRAPHING PACKAGE

1 The weight W_t of bacteria in a culture t hours after establishment is given by $W_t = 100 \times 2^{0.1t}$ grams.

a Find the initial weight.

b Find the weight after: **i** 4 hours **ii** 10 hours **iii** 24 hours.

c Sketch the graph of W_t against t using the results of **a** and **b** only.

d Use technology to graph $Y_1 = 100 \times 2^{0.1X}$ and check your answers to **a**, **b**, and **c**.

2 A breeding program to ensure the survival of pygmy possums is established with an initial population of 50 (25 pairs). From a previous program, the expected population P_n in n years' time is given by $P_n = P_0 \times 2^{0.3n}$.

a What is the value of P_0?

b What is the expected population after: **i** 2 years **ii** 5 years **iii** 10 years?

c Sketch the graph of P_n against n using **a** and **b** only.

d Use technology to graph $Y_1 = 50 \times 2^{0.3X}$ and check your answers to **b**.

3 A species of bear is introduced to a large island off Alaska where previously there were no bears. 6 pairs of bears were introduced in 1998. It is expected that the population will increase according to $B_t = B_0 \times 2^{0.18t}$ where t is the time since the introduction.

a Find B_0. **b** Find the expected bear population in 2018.

c Find the expected percentage increase from 2008 to 2018.

4 The speed V_t of a chemical reaction is given by $V_t = V_0 \times 2^{0.05t}$ where t is the temperature in °C.

a Find the reaction speed at: **i** 0°C **ii** 20°C.

b Find the percentage increase in reaction speed at 20°C compared with 0°C.

c Find $\left(\dfrac{V_{50} - V_{20}}{V_{20}}\right) \times 100\%$ and explain what this calculation means.

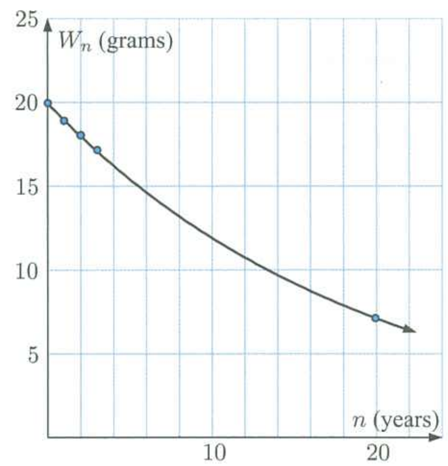
DECAY

Consider a radioactive substance with original weight 20 grams. It *decays* or reduces by 5% each year. The multiplier for this is 95% or 0.95.

If W_n is the weight after n years, then:

$W_0 = 20$ grams
$W_1 = W_0 \times 0.95 = 20 \times 0.95$ grams
$W_2 = W_1 \times 0.95 = 20 \times (0.95)^2$ grams
$W_3 = W_2 \times 0.95 = 20 \times (0.95)^3$ grams
$\vdots$
$W_{20} = 20 \times (0.95)^{20} \approx 7.2$ grams
$\vdots$
$W_{100} = 20 \times (0.95)^{100} \approx 0.1$ grams

and from this pattern we see that $W_n = 20 \times (0.95)^n$.



Example 22 Self Tutor

When a diesel-electric generator is switched off, the current dies away according to the formula $I(t) = 24 \times (0.25)^t$ amps, where t is the time in seconds after the power is cut.

a Find $I(t)$ when $t = 0$, 1, 2 and 3.

b What current flowed in the generator at the instant when it was switched off?

c Plot the graph of $I(t)$ for $t \geqslant 0$ using the information above.

d Use your graph or technology to find how long it takes for the current to reach 4 amps.

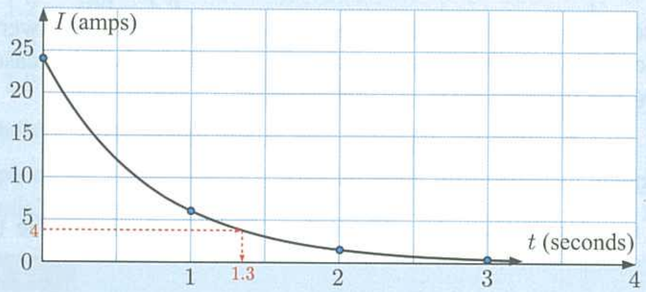
a $I(t) = 24 \times (0.25)^t$ amps

$I(0)$	$I(1)$	$I(2)$	$I(3)$
$= 24 \times (0.25)^0$	$= 24 \times (0.25)^1$	$= 24 \times (0.25)^2$	$= 24 \times (0.25)^3$
$= 24$ amps	$= 6$ amps	$= 1.5$ amps	$= 0.375$ amps

b When $t = 0$, $I(0) = 24$

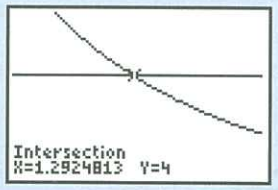
When the generator was switched off, 24 amps of current flowed in the circuit.

c



d From the graph above, the time to reach 4 amps is about 1.3 seconds. *or*

By finding the **point of intersection** of $Y_1 = 24 \times (0.25)$^X and $Y_2 = 4$ on a graphics calculator, the solution is ≈ 1.29 seconds.



Example 23 Self Tutor

The weight of radioactive material remaining after t years is given by $W_t = W_0 \times 2^{-0.001t}$ grams.

a Find the original weight.

b Find the percentage remaining after 200 years.

a When $t = 0$, the weight remaining is $W_0 \times 2^0 = W_0$

$\therefore$ W_0 is the original weight.

b When $t = 200$,
$$\begin{aligned} W_{200} &= W_0 \times 2^{-0.001\times 200} \\ &= W_0 \times 2^{-0.2} \\ &\approx W_0 \times 0.8706 \\ &\approx 87.06\% \text{ of } W_0 \end{aligned}$$

After 200 years, 87.1% of the material remains.

EXERCISE 3G.2

1 The weight of a radioactive substance t years after being set aside is given by $W(t) = 250 \times (0.998)^t$ grams.

a How much radioactive substance was initially set aside?

b Determine the weight of the substance after:

i 400 years **ii** 800 years **iii** 1200 years.

c Sketch the graph of $W(t)$ for $t \geqslant 0$ using **a** and **b** only.

d Use your graph or graphics calculator to find how long it takes for the substance to decay to 125 grams.

2 The temperature T of a liquid which has been placed in a refrigerator is given by $T(t) = 100 \times 2^{-0.02t}$ °C where t is the time in minutes.

a Find the initial temperature of the liquid.

b Find the temperature after:

i 15 minutes **ii** 20 minutes **iii** 78 minutes.

c Sketch the graph of $T(t)$ for $t \geqslant 0$ using **a** and **b** only.

3 Answer the **Opening Problem** on page **82**.

4 The weight W_t grams of radioactive substance remaining after t years is given by $W_t = 1000 \times 2^{-0.03t}$ grams.

a Find the initial weight of the radioactive substance.

b Find the weight remaining after:

i 10 years **ii** 100 years **iii** 1000 years.

c Graph W_t against t using **a** and **b** only.

d Use your graph or graphics calculator to find the time when 10 grams of the substance remains.

e Write an expression for the amount of substance that has decayed after t years.

5 The weight W_t of a radioactive uranium-235 sample remaining after t years is given by the formula $W_t = W_0 \times 2^{-0.0002t}$ grams, $t \geqslant 0$. Find:

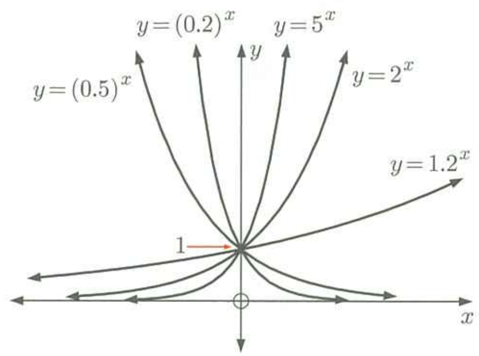
a the original weight **b** the percentage weight loss after 1000 years

c the time required until $\frac{1}{512}$ of the sample remains.

H THE NATURAL EXPONENTIAL e^x

We have seen that the simplest exponential functions are of the form $f(x) = b^x$ where $b > 0$, $b \neq 1$.

Graphs of some of these functions are shown alongside.



We can see that for all positive values of the base b, the graph is always positive.

Hence $b^x > 0$ for all $b > 0$.

There are an infinite number of possible choices for the base number.

However, where exponential data is examined in science, engineering, and finance, the base $e \approx 2.7183$ is commonly used.

e is a special number in mathematics. It is irrational like π, and just as π is the ratio of a circle's circumference to its diameter, e also has a physical meaning. We explore this meaning in the following investigation.

INVESTIGATION 2 CONTINUOUS COMPOUND INTEREST

A formula for calculating the amount to which an investment grows is $u_n = u_0(1+i)^n$ where:

u_n is the final amount, u_0 is the initial amount,

i is the interest rate per compounding period,

n is the number of periods or number of times the interest is compounded.

We will investigate the final value of an investment for various values of n, and allow n to get extremely large.

What to do:

1 Suppose \$1000 is invested for one year at a fixed rate of 6% per annum. Use your calculator to find the final amount or *maturing value* if the interest is paid:

a annually $(n = 1,\ i = 6\% = 0.06)$ **b** quarterly $(n = 4,\ i = \frac{6\%}{4} = 0.015)$

c monthly **d** daily

e by the second **f** by the millisecond.

2 Comment on your answers from **1**.

3 If r is the percentage rate per year, t is the number of years, and N is the number of interest payments per year, then $i = \frac{r}{N}$ and $n = Nt$.

The growth formula becomes $u_n = u_0\left(1+\frac{r}{N}\right)^{Nt}$.

If we let $a = \frac{N}{r}$, show that $u_n = u_0\left[\left(1+\frac{1}{a}\right)^a\right]^{rt}$.

4 For continuous compound growth, the number of interest payments per year N gets very large.

a Explain why a gets very large as N gets very large.

b Copy and complete the table, giving your answers as accurately as technology permits.

a	$\left(1+\frac{1}{a}\right)^a$
10	
100	
1000	
10 000	
100 000	
1 000 000	
10 000 000	

5 You should have found that for very large values of a,

$$\left(1+\frac{1}{a}\right)^a \approx 2.718\,281\,828\,459....$$

Use the $\boxed{e^x}$ key of your calculator to find the value of e^1. What do you notice?

6 For continuous growth, $u_n = u_0e^{rt}$ where u_0 is the initial amount

r is the annual percentage rate

t is the number of years

Use this formula to find the final value if \$1000 is invested for 4 years at a fixed rate of 6% per annum, where the interest is calculated continuously.

From **Investigation 2** we observe that:

If interest is paid *continuously* or *instantaneously* then the formula for calculating a compounding amount $u_n = u_0(1+i)^n$ can be replaced by $u_n = u_0e^{rt}$, where r is the percentage rate per annum and t is the number of years.

HISTORICAL NOTE

The natural exponential e was first described in 1683 by Swiss mathematician Jacob Bernoulli. He discovered the number while studying compound interest, just as we did in **Investigation 2**.

The natural exponential was first called e by Swiss mathematician and physicist Leonhard Euler in a letter to the German mathematician Christian Goldbach in 1731. The number was then published with this notation in 1736.

In 1748 Euler evaluated e correct to 18 decimal places.

One may think that e was chosen because it was the first letter of Euler's name or for the word exponential, but it is likely that it was just the next vowel available since he had already used a in his work.

Leonhard Euler

EXERCISE 3H

GRAPHING PACKAGE

1 Sketch, on the same set of axes, the graphs of $y = 2^x$, $y = e^x$, and $y = 3^x$. Comment on any observations.

2 Sketch, on the same set of axes, the graphs of $y = e^x$ and $y = e^{-x}$. What is the geometric connection between these two graphs?

3 For the general exponential function $y = ae^{kx}$, what is the y-intercept?

4 Consider $y = 2e^x$.

a Explain why y can never be < 0. **b** Find y if: **i** $x = -20$ **ii** $x = 20$.

5 Find, to 3 significant figures, the value of:

a e^2 **b** e^3 **c** $e^{0.7}$ **d** $\sqrt{e}$ **e** e^{-1}

6 Write the following as powers of e:

a $\sqrt{e}$ **b** $\frac{1}{\sqrt{e}}$ **c** $\frac{1}{e^2}$ **d** $e\sqrt{e}$

7 Simplify:

a $(e^{0.36})^{\frac{t}{2}}$ **b** $(e^{0.064})^{\frac{t}{16}}$ **c** $(e^{-0.04})^{\frac{t}{8}}$ **d** $(e^{-0.836})^{\frac{t}{5}}$

8 Find, to five significant figures, the values of:

a $e^{2.31}$ **b** $e^{-2.31}$ **c** $e^{4.829}$

d $e^{-4.829}$ **e** $50e^{-0.1764}$ **f** $80e^{-0.6342}$

g $1000e^{1.2642}$ **h** $0.25e^{-3.6742}$

9 On the same set of axes, sketch and clearly label the graphs of:
$f : x \mapsto e^x$, $\quad g : x \mapsto e^{x-2}$, $\quad h : x \mapsto e^x + 3$
State the domain and range of each function.

10 On the same set of axes, sketch and clearly label the graphs of:
$f : x \mapsto e^x$, $\quad g : x \mapsto -e^x$, $\quad h : x \mapsto 10 - e^x$
State the domain and range of each function.

11 Expand and simplify:

a $(e^x + 1)^2$ **b** $(1 + e^x)(1 - e^x)$ **c** $e^x(e^{-x} - 3)$

12 The weight of bacteria in a culture is given by $W(t) = 2e^{\frac{t}{2}}$ grams where t is the time in hours after the culture was set to grow.

a Find the weight of the culture when:
i $t = 0$ **ii** $t = 30$ min **iii** $t = 1\frac{1}{2}$ hours **iv** $t = 6$ hours.

b Use **a** to sketch the graph of $W(t) = 2e^{\frac{t}{2}}$.

13 Solve for x:

a $e^x = \sqrt{e}$ **b** $e^{\frac{1}{2}x} = \frac{1}{e^2}$

14 The current flowing in an electrical circuit t seconds after it is switched off is given by $I(t) = 75e^{-0.15t}$ amps.

a What current is still flowing in the circuit after:
i 1 second **ii** 10 seconds?

b Use your graphics calculator to sketch $I(t) = 75e^{-0.15t}$ and $I = 1$.

c Hence find how long it will take for the current to fall to 1 amp.

15 Consider the function $f(x) = e^x$.

a On the same set of axes, sketch $y = f(x)$, $y = x$, and $y = f^{-1}(x)$.

b State the domain and range of f^{-1}.

ACTIVITY

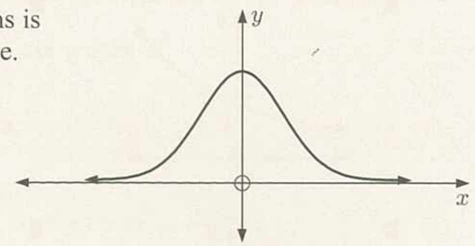
Click on the icon to run a card game for exponential functions.

CARD GAME

RESEARCH — RESEARCHING *e*

What to do:

1 The 'bell curve' which models statistical distributions is shown alongside. Research the equation of this curve.



2 The function $f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{2\times 3}x^3 + \frac{1}{2\times 3\times 4}x^4 + ...$ has infinitely many terms.
It can be shown that $f(x) = e^x$.
Check this statement by finding an approximation for $f(1)$ using its first 20 terms.

REVIEW SET 3A — NON-CALCULATOR

1 Simplify:

a $-(-1)^{10}$ **b** $-(-3)^3$ **c** $3^0 - 3^{-1}$

2 Simplify using the laws of exponents:

a $a^4b^5 \times a^2b^2$ **b** $6xy^5 \div 9x^2y^5$ **c** $\frac{5(x^2y)^2}{(5x^2)^2}$

3 Let $f(x) = 3^x$.

a Write down the value of: **i** $f(4)$ **ii** $f(-1)$

b Find the value of k such that $f(x+2) = k\,f(x)$, $k \in \mathbb{Z}$.

4 Write without brackets or negative exponents:

a $x^{-2} \times x^{-3}$ **b** $2(ab)^{-2}$ **c** $2ab^{-2}$

5 Write as a single power of 3:

a $\frac{27}{9^a}$ **b** $(\sqrt{3})^{1-x} \times 9^{1-2x}$

6 Evaluate:

a $8^{\frac{2}{3}}$ **b** $27^{-\frac{2}{3}}$

7 Write without negative exponents:

a mn^{-2} **b** $(mn)^{-3}$ **c** $\frac{m^2n^{-1}}{p^{-2}}$ **d** $(4m^{-1}n)^2$

8 Expand and simplify:

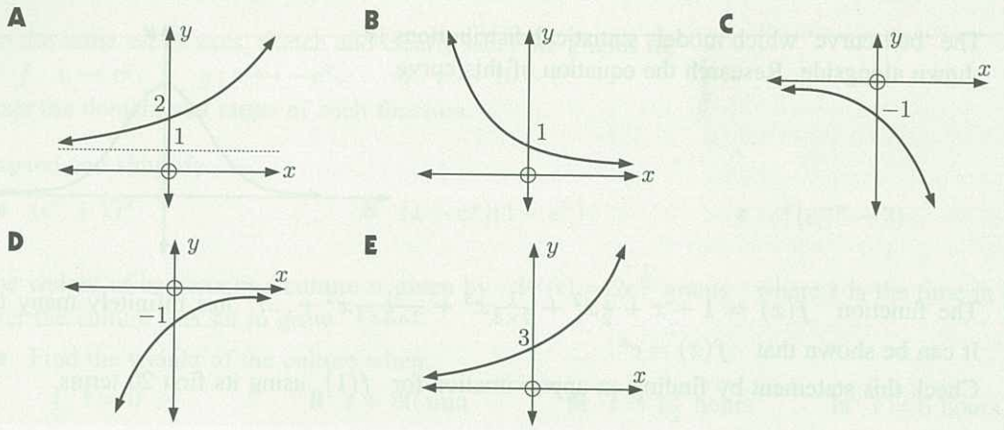
a $(3 - e^x)^2$ **b** $(\sqrt{x} + 2)(\sqrt{x} - 2)$ **c** $2^{-x}(2^{2x} + 2^x)$

9 Find the value of x:

a $2^{x-3} = \frac{1}{32}$ **b** $9^x = 27^{2-2x}$ **c** $e^{2x} = \frac{1}{\sqrt{e}}$

10 Match each equation to its corresponding graph:

a $y = -e^x$ **b** $y = 3 \times 2^x$ **c** $y = e^x + 1$ **d** $y = 3^{-x}$ **e** $y = -e^{-x}$



11 Suppose $y = a^x$. Express in terms of y:

a a^{2x} **b** a^{-x} **c** $\dfrac{1}{\sqrt{a^x}}$

REVIEW SET 3B CALCULATOR

1 **a** Write 4×2^n as a power of 2.

b Evaluate $7^{-1} - 7^0$.

c Write $(\frac{2}{3})^{-3}$ in simplest fractional form.

d Write $\left(\dfrac{2a^{-1}}{b^2}\right)^2$ without negative exponents or brackets.

2 Evaluate, correct to 3 significant figures:

a $3^{\frac{3}{4}}$ **b** $27^{-\frac{1}{5}}$ **c** $\sqrt[4]{100}$

3 If $f(x) = 3 \times 2^x$, find the value of:

a $f(0)$ **b** $f(3)$ **c** $f(-2)$

4 Suppose $f(x) = 2^{-x} + 1$.

a Find $f(\frac{1}{2})$. **b** Find a such that $f(a) = 3$.

5 On the same set of axes draw the graphs of $y = 2^x$ and $y = 2^x - 4$. Include on your graph the y-intercept and the equation of the horizontal asymptote of each function.

6 The temperature of a dish t minutes after it is removed from the microwave, is given by $T = 80 \times (0.913)^t$ °C.

a Find the initial temperature of the dish.

b Find the temperature after: **i** $t = 12$ **ii** $t = 24$ **iii** $t = 36$ minutes.

c Draw the graph of T against t for $t \geqslant 0$, using the above or technology.

d Hence, find the time taken for the temperature of the dish to fall to 25°C.

7 Consider $y = 3^x - 5$.

a Find y when $x = 0, \pm 1, \pm 2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3^x - 5$.

d State the equation of any asymptote.

8 **a** On the same set of axes, sketch and clearly label the graphs of:

$f : x \mapsto e^x$, $g : x \mapsto e^{x-1}$, $h : x \mapsto 3 - e^x$

b State the domain and range of each function in **a**.

9 Consider $y = 3 - 2^{-x}$.

a Find y when $x = 0, \pm 1, \pm 2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 3 - 2^{-x}$.

d State the equation of any asymptote.

10 The weight of a radioactive substance after t years is given by $W = 1500 \times (0.993)^t$ grams.

a Find the original amount of radioactive material.

b Find the amount of radioactive material remaining after:

i 400 years **ii** 800 years.

c Sketch the graph of W against t, $t \geqslant 0$, using the above or technology.

d Hence, find the time taken for the weight to reduce to 100 grams.

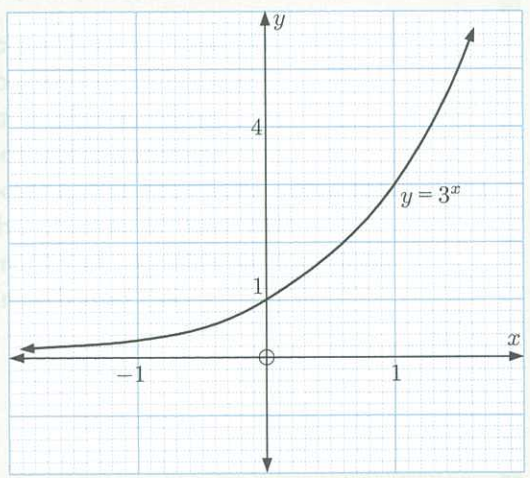
REVIEW SET 3C

1 Given the graph of $y = 3^x$ shown, estimate solutions to the exponential equations:

a $3^x = 5$

b $3^x = \frac{1}{2}$

c $6 \times 3^x = 20$



2 Simplify using the laws of exponents:

a $(a^7)^3$ **b** $pq^2 \times p^3q^4$ **c** $\dfrac{8ab^5}{2a^4b^4}$

3 Write the following as a power of 2:

a 2×2^{-4} **b** $16 \div 2^{-3}$ **c** 8^4

4 Write without brackets or negative exponents:

a b^{-3} **b** $(ab)^{-1}$ **c** ab^{-1}

5 Simplify $\dfrac{2^{x+1}}{2^{1-x}}$.

6 Write as powers of 5 in simplest form:

a 1 **b** $5\sqrt{5}$ **c** $\dfrac{1}{\sqrt[4]{5}}$ **d** 25^{a+3}

7 Expand and simplify:

a $e^x(e^{-x}+e^x)$ **b** $(2^x+5)^2$ **c** $(x^{\frac{1}{2}}-7)(x^{\frac{1}{2}}+7)$

8 Solve for x:

a $6 \times 2^x = 192$ **b** $4 \times (\frac{1}{3})^x = 324$

9 The point $(1, \sqrt{8})$ lies on the graph of $y = 2^{kx}$. Find the value of k.

10 Solve for x without using a calculator:

a $2^{x+1} = 32$ **b** $4^{x+1} = \left(\frac{1}{8}\right)^x$

11 Consider $y = 2e^{-x} + 1$.

a Find y when $x = 0, \pm1, \pm2$.

b Discuss y as $x \to \pm\infty$.

c Sketch the graph of $y = 2e^{-x} + 1$.

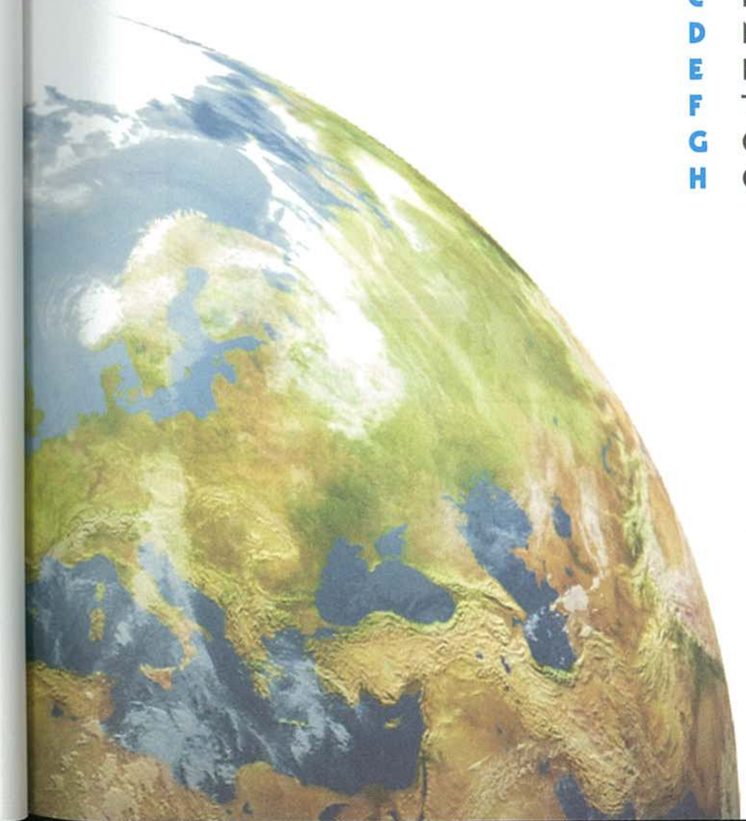
d State the equation of any asymptote.

Chapter 4

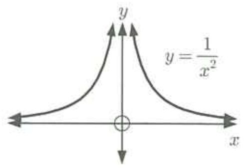
Logarithms

Syllabus reference: 1.2, 2.2, 2.6, 2.8

Contents:



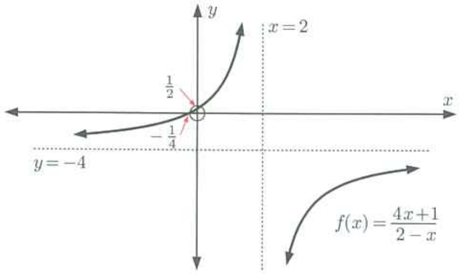
4 **a** $x = 0$ **b**



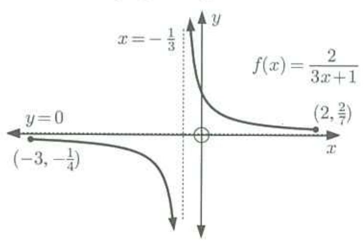
c Domain $= \{x \mid x \neq 0\}$, Range $= \{y \mid y > 0\}$

5 **a** $a = 2,\ b = -1$
b Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq -1\}$

6 **a** vertical asymptote $x = 2$, horizontal asymptote $y = -4$
b Domain $= \{x \mid x \neq 2\}$, Range $= \{y \mid y \neq -4\}$
c as $x \to 2^-$, $y \to \infty$ as $x \to \infty$, $y \to -4^-$
as $x \to 2^+$, $y \to -\infty$ as $x \to -\infty$, $y \to -4^+$
d x-intercept $-\frac{1}{4}$, y-intercept $\frac{1}{2}$
e

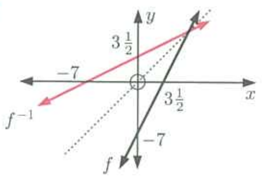


7 **a** $(g \circ f)(x) = \dfrac{2}{3x+1}$ **b** $x = -\frac{1}{2}$
c **i** vertical asymptote $x = -\frac{1}{3}$,
horizontal asymptote $y = 0$
ii



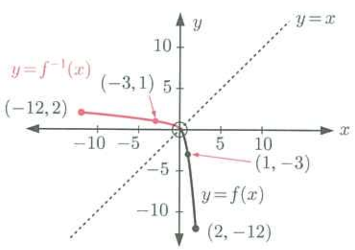
iii Range $= \{y \mid y \leqslant -\frac{1}{4}$ or $y \geqslant \frac{2}{7}\}$

8 **a**



b $f^{-1}(x) = \dfrac{x+7}{2}$

9 **a**



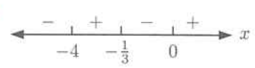
b Range $= \{y \mid 0 \leqslant y \leqslant 2\}$
c **i** $x \approx -1.83$ **ii** $x = -3$

REVIEW SET 2C

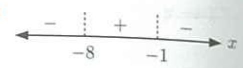
1 **a** Domain $= \{x \mid x \geqslant -2\}$, Range $= \{y \mid 1 \leqslant y < 3\}$
b Domain $= \{x \in \mathbb{R}\}$, Range $= \{y \mid y = -1, 1 \text{ or } 2\}$

2 **a** 12 **b** $x = \pm 1$ **3** **a** $x = \frac{1}{2}$ **b** $x < -7$

4 **a**



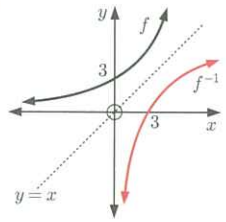
b



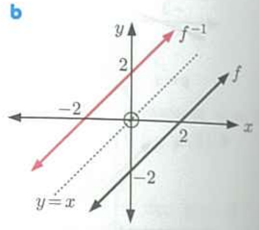
5 **a** $10 - 6x$ **b** $x = 2$ **6** **a** $1 - 2\sqrt{x}$ **b** $\sqrt{1 - 2x}$

7 $a = 1,\ b = -6,\ c = 5$

8 **a**



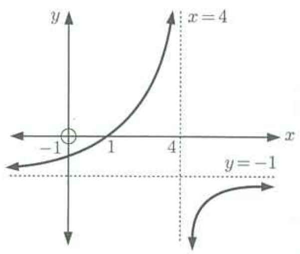
b



9 **a** $f^{-1}(x) = \dfrac{7-x}{4}$ **b** $f^{-1}(x) = \dfrac{5x-3}{2}$

10 $(f^{-1} \circ h^{-1})(x) = (h \circ f)^{-1}(x) = \dfrac{4x+6}{15}$ **11** 16

12



EXERCISE 3A

1 **a** $2^1 = 2,\ 2^2 = 4,\ 2^3 = 8,\ 2^4 = 16,\ 2^5 = 32,\ 2^6 = 64$
b $3^1 = 3,\ 3^2 = 9,\ 3^3 = 27,\ 3^4 = 81,\ 3^5 = 243,$
$3^6 = 729$
c $4^1 = 4,\ 4^2 = 16,\ 4^3 = 64,\ 4^4 = 256,\ 4^5 = 1024,$
$4^6 = 4096$

2 **a** $5^1 = 5,\ 5^2 = 25,\ 5^3 = 125,\ 5^4 = 625$
b $6^1 = 6,\ 6^2 = 36,\ 6^3 = 216,\ 6^4 = 1296$
c $7^1 = 7,\ 7^2 = 49,\ 7^3 = 343,\ 7^4 = 2401$

3 **a** -1 **b** 1 **c** 1 **d** -1 **e** 1 **f** -1
g -1 **h** -32 **i** -32 **j** -64 **k** 625 **l** -625

4 **a** 16 384 **b** 2401 **c** -3125 **d** -3125
e 262 144 **f** 262 144 **g** $-262\,144$
h 902.436 039 6 **i** $-902.436\,039\,6$
j $-902.436\,039\,6$

5 **a** $0.\overline{1}$ **b** $0.\overline{1}$ **c** $0.02\overline{7}$ **d** $0.02\overline{7}$
e 0.012 345 679 **f** 0.012 345 679 **g** 1 **h** 1
Notice that $a^{-n} = \dfrac{1}{a^n}$

6 3 **7** 7

EXERCISE 3B

1 **a** 5^{11} **b** d^8 **c** k^5 **d** $\frac{1}{7}$ **e** x^{10} **f** 3^{16}
g p^{-4} **h** n^{12} **i** 5^{3t} **j** 7^{x+2} **k** 10^{3-q} **l** c^{4m}

2 **a** 2^2 **b** 2^{-2} **c** 2^3 **d** 2^{-3} **e** 2^5 **f** 2^{-5}
g 2^1 **h** 2^{-1} **i** 2^6 **j** 2^{-6} **k** 2^7 **l** 2^{-7}

3 **a** 3^2 **b** 3^{-2} **c** 3^3 **d** 3^{-3} **e** 3^1 **f** 3^{-1}
g 3^4 **h** 3^{-4} **i** 3^0 **j** 3^5 **k** 3^{-5}

4 **a** 2^{a+1} **b** 2^{b+2} **c** 2^{t+3} **d** 2^{2x+2} **e** 2^{n-1}
f 2^{c-2} **g** 2^{2m} **h** 2^{n+1} **i** 2^1 **j** 2^{3x-1}

5 **a** 3^{p+2} **b** 3^{3a} **c** 3^{2n+1} **d** 3^{d+3} **e** 3^{3t+2}
f 3^{y-1} **g** 3^{1-y} **h** 3^{2-3t} **i** 3^{3a-1} **j** 3^3

6 **a** $4a^2$ **b** $27b^3$ **c** a^4b^4 **d** p^3q^3 **e** $\dfrac{m^2}{n^2}$
f $\dfrac{a^3}{27}$ **g** $\dfrac{b^4}{c^4}$ **h** $1,\ b \neq 0$ **i** $\dfrac{m^4}{81n^4}$ **j** $\dfrac{x^3y^3}{8}$

7 **a** $4a^2$ **b** $36b^4$ **c** $-8a^3$ **d** $-27m^6n^6$
e $16a^4b^{16}$ **f** $\dfrac{-8a^6}{b^6}$ **g** $\dfrac{16a^6}{b^2}$ **h** $\dfrac{9p^4}{q^6}$

8 **a** $\dfrac{a}{b^2}$ **b** $\dfrac{1}{a^2b^2}$ **c** $\dfrac{4a^2}{b^2}$ **d** $\dfrac{9b^2}{a^4}$ **e** $\dfrac{a^2}{bc^2}$
f $\dfrac{a^2c^2}{b}$ **g** a^3 **h** $\dfrac{b^3}{a^2}$ **i** $\dfrac{2}{ad^2}$ **j** $12am^3$

9 **a** a^{-n} **b** b^n **c** 3^{n-2} **d** a^nb^m **e** a^{-2n-2}

10 **a** 1 **b** $\frac{4}{7}$ **c** 6 **d** 27 **e** $\frac{9}{16}$ **f** $\frac{5}{2}$
g $\frac{27}{125}$ **h** $\frac{151}{5}$

11 **a** 3^{-2} **b** 2^{-4} **c** 5^{-3} **d** $3^1 \times 5^{-1}$ **e** $2^2 \times 3^{-3}$
f $2^{c-3} \times 3^{-2}$ **g** $3^{2k} \times 2^{-1} \times 5^{-1}$ **h** $2^p \times 3^{p-1} \times 5^{-2}$

12 **a** $5^3 = 21 + 23 + 25 + 27 + 29$
b $7^3 = 43 + 45 + 47 + 49 + 51 + 53 + 55$
c $12^3 = 133 + 135 + 137 + 139 + 141 + 143 + 145 + 147$
$+ 149 + 151 + 153 + 155$

EXERCISE 3C

1 **a** $2^{\frac{1}{5}}$ **b** $2^{-\frac{1}{5}}$ **c** $2^{\frac{3}{2}}$ **d** $2^{\frac{5}{2}}$ **e** $2^{-\frac{1}{3}}$
f $2^{\frac{4}{3}}$ **g** $2^{\frac{3}{2}}$ **h** $2^{\frac{3}{2}}$ **i** $2^{-\frac{4}{3}}$ **j** $2^{-\frac{3}{2}}$

2 **a** $3^{\frac{1}{3}}$ **b** $3^{-\frac{1}{3}}$ **c** $3^{\frac{1}{4}}$ **d** $3^{\frac{3}{2}}$ **e** $3^{-\frac{5}{2}}$

3 **a** $7^{\frac{1}{3}}$ **b** $3^{\frac{3}{4}}$ **c** $2^{\frac{4}{5}}$ **d** $2^{\frac{5}{3}}$ **e** $7^{\frac{2}{7}}$
f $7^{-\frac{1}{3}}$ **g** $3^{-\frac{3}{4}}$ **h** $2^{-\frac{4}{5}}$ **i** $2^{-\frac{5}{3}}$ **j** $7^{-\frac{2}{7}}$

4 **a** 2.28 **b** 1.83 **c** 0.794 **d** 0.435 **e** 1.68
f 1.93 **g** 0.523

5 **a** 8 **b** 32 **c** 8 **d** 125 **e** 4
f $\frac{1}{2}$ **g** $\frac{1}{27}$ **h** $\frac{1}{16}$ **i** $\frac{1}{81}$ **j** $\frac{1}{25}$

EXERCISE 3D.1

1 **a** $x^5 + 2x^4 + x^2$ **b** $4^x + 2^x$ **c** $x + 1$
d $49^x + 2(7^x)$ **e** $2(3^x) - 1$ **f** $x^2 + 2x + 3$
g $1 + 5(2^{-x})$ **h** $5^x + 1$ **i** $x^{\frac{3}{2}} + x^{\frac{1}{2}} + 1$

2 **a** $4^x + 2^{x+1} - 3$ **b** $9^x + 7(3^x) + 10$
c $25^x - 6(5^x) + 8$ **d** $4^x + 6(2^x) + 9$
e $9^x - 2(3^x) + 1$ **f** $16^x + 14(4^x) + 49$
g $x - 4$ **h** $4^x - 9$ **i** $x - x^{-1}$ **j** $x^2 + 4 + \dfrac{4}{x^2}$
k $7^{2x} - 2 + 7^{-2x}$ **l** $25 - 10(2^{-x}) + 4^{-x}$

EXERCISE 3D.2

1 **a** $5^x(5^x + 1)$ **b** $10(3^n)$ **c** $7^n(1 + 7^{2n})$
d $5(5^n - 1)$ **e** $6(6^{n+1} - 1)$ **f** $16(4^n - 1)$

2 **a** $(3^x+2)(3^x-2)$ **b** $(2^x+5)(2^x-5)$ **c** $(4+3^x)(4-3^x)$
d $(5 + 2^x)(5 - 2^x)$ **e** $(3^x + 2^x)(3^x - 2^x)$ **f** $(2^x + 3)^2$
g $(3^x + 5)^2$ **h** $(2^x - 7)^2$ **i** $(5^x - 2)^2$

3 **a** $(2^x + 3)(2^x + 6)$ **b** $(2^x + 4)(2^x - 5)$
c $(3^x + 2)(3^x + 7)$ **d** $(3^x + 5)(3^x - 1)$
e $(5^x + 2)(5^x - 1)$ **f** $(7^x - 4)(7^x - 3)$

4 **a** 2^n **b** 10^a **c** 3^b **d** $\dfrac{1}{5^n}$ **e** 5^x
f $(\frac{3}{4})^a$ **g** 5 **h** 5^n

5 **a** $3^m + 1$ **b** $1 + 6^n$ **c** $4^n + 2^n$ **d** $4^x - 1$
e 6^n **f** 5^n **g** 4 **h** $2^n - 1$ **i** $\frac{1}{2}$

6 **a** $n\,2^{n+1}$ **b** -3^{n-1}

EXERCISE 3E

1 **a** $x = 3$ **b** $x = 2$ **c** $x = 4$ **d** $x = 0$
e $x = -1$ **f** $x = \frac{1}{2}$ **g** $x = -3$ **h** $x = 2$
i $x = -3$ **j** $x = -4$ **k** $x = 2$ **l** $x = 1$

2 **a** $x = \frac{5}{3}$ **b** $x = -\frac{3}{2}$ **c** $x = -\frac{3}{2}$ **d** $x = -\frac{1}{2}$
e $x = -\frac{2}{3}$ **f** $x = -\frac{5}{4}$ **g** $x = \frac{3}{2}$ **h** $x = \frac{5}{2}$
i $x = \frac{1}{8}$ **j** $x = \frac{9}{2}$ **k** $x = -4$ **l** $x = -4$
m $x = 0$ **n** $x = \frac{7}{2}$ **o** $x = -2$ **p** $x = -6$

3 **a** $x = \frac{1}{7}$ **b** has no solutions **c** $x = 2\frac{1}{2}$

4 **a** $x = 3$ **b** $x = 3$ **c** $x = 2$
d $x = 2$ **e** $x = -2$ **f** $x = -2$

5 **a** $x = 1$ or 2 **b** $x = 1$ **c** $x = 1$ or 2
d $x = 1$ **e** $x = 2$ **f** $x = 0$

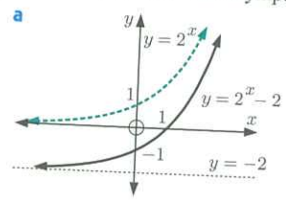
EXERCISE 3F

1 **a** 1.4 **b** 1.7 **c** 2.8 **d** 0.4

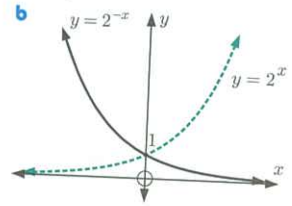
2 **a** $x \approx 1.6$ **b** $x \approx -0.7$

3 $y = 2^x$ has a horizontal asymptote of $y = 0$

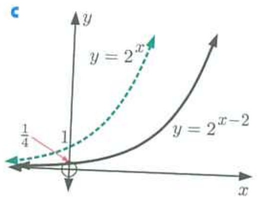
4 **a**



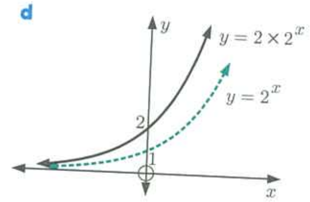
b



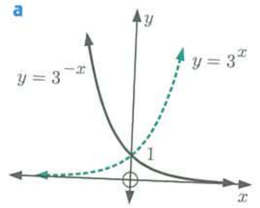
c



d



5 **a**



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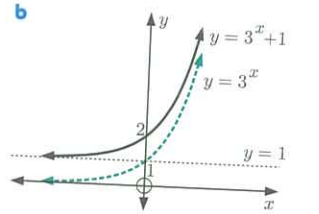
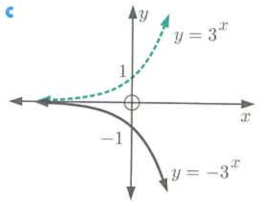
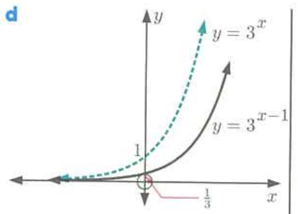


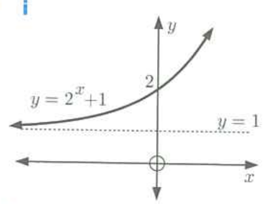
c



d



6 **a** **i**



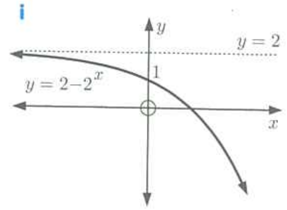
ii Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y > 1\}$

iii $y \approx 3.67$

iv As $x \to \infty$, $y \to \infty$
As $x \to -\infty$, $y \to 1$ from above

v $y = 1$

b **i**



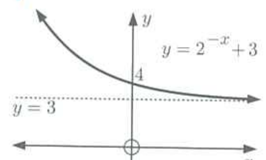
ii Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y < 2\}$

iii $y \approx -0.665$

iv As $x \to \infty$, $y \to -\infty$
As $x \to -\infty$, $y \to 2$ from below

v $y = 2$

c **i**



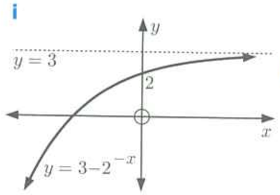
ii Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y > 3\}$

iii $y \approx 3.38$

iv As $x \to \infty$, $y \to 3$ from above
As $x \to -\infty$, $y \to \infty$

v $y = 3$

d **i**



ii Domain: $\{x \mid x \in \mathbb{R}\}$
Range: $\{y \mid y < 3\}$

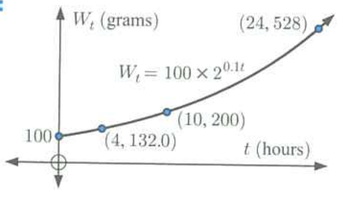
iii $y \approx 2.63$

iv As $x \to \infty$, $y \to 3$ from below
As $x \to -\infty$, $y \to -\infty$

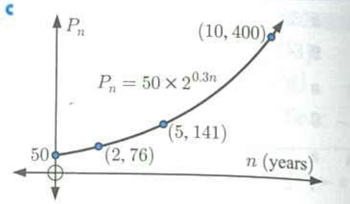
v $y = 3$

EXERCISE 3G.1

1 **a** 100 grams
b **i** 132 g **ii** 200 g **iii** 528 g
c



2 **a** 50
b **i** 76 **ii** 141 **iii** 400
c

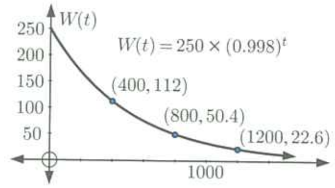


3 **a** 12 bears **b** 146 bears **c** 248% increase

4 **a** **i** V_0 **ii** $2V_0$ **b** 100%
c 183% increase, it is the percentage increase at 50°C compared with 20°C

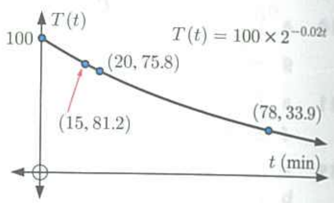
EXERCISE 3G.2

1 **a** 250 g **b** **i** 112 g **ii** 50.4 g **iii** 22.6 g
c

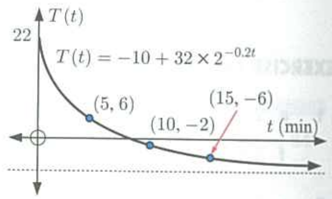


d ≈ 346 years

2 **a** 100°C
b **i** 81.2°C **ii** 75.8°C **iii** 33.9°C
c

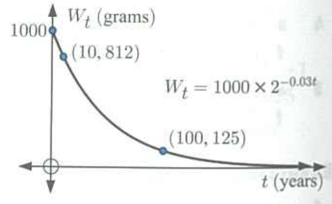


3 **a** **i** 22°C **ii** 6°C **iii** -2°C **iv** -6°C
b



c Never, since $32 \times 2^{-0.2t}$ is always > 0

4 **a** 1000 g
b **i** 812 g **ii** 125 g **iii** 9.31×10^{-7} g
c

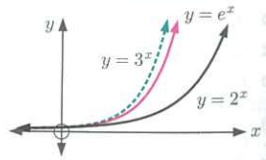


d 221 years
e $1000(1 - 2^{-0.03t})$ g

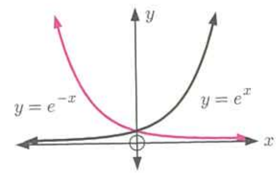
5 **a** W_0 **b** 12.9% **c** 45 000 years

EXERCISE 3H

1 The graph of $y = e^x$ lies between $y = 2^x$ and $y = 3^x$.



2 One is the other reflected in the y-axis.



3 *a*

4 **a** $e^x > 0$ for all x
b **i** 0.000 000 004 12 **ii** 970 000 000

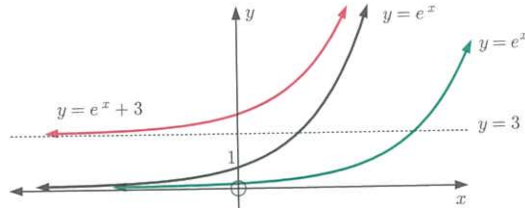
5 **a** ≈ 7.39 **b** ≈ 20.1 **c** ≈ 2.01 **d** ≈ 1.65
e ≈ 0.368

6 **a** $e^{\frac{1}{2}}$ **b** $e^{-\frac{1}{2}}$ **c** e^{-2} **d** $e^{\frac{3}{2}}$

7 **a** $e^{0.18t}$ **b** $e^{0.004t}$ **c** $e^{-0.005t}$ **d** $\approx e^{-0.167t}$

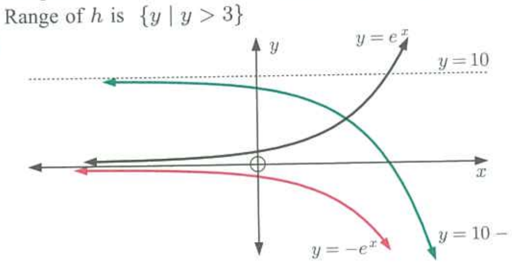
8 **a** 10.074 **b** 0.099 261 **c** 125.09 **d** 0.007 994 5
e 41.914 **f** 42.429 **g** 3540.3 **h** 0.006 342 4

9



Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y > 0\}$
Range of h is $\{y \mid y > 3\}$

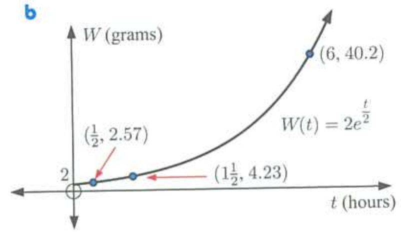
10



Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y < 0\}$
Range of h is $\{y \mid y < 10\}$

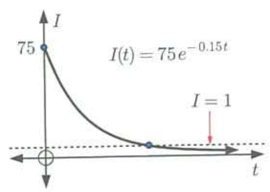
11 **a** $e^{2x} + 2e^x + 1$ **b** $1 - e^{2x}$ **c** $1 - 3e^x$

12 **a** **i** 2 g **ii** 2.57 g **iii** 4.23 g **iv** 40.2 g
b



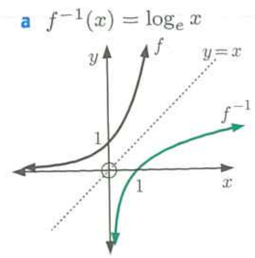
13 **a** $x = \frac{1}{2}$ **b** $x = -4$

14 **a** **i** 64.6 amps **ii** 16.7 amps
b



c 28.8 seconds

15 **a** $f^{-1}(x) = \log_e x$



b Domain of f^{-1} is $\{x \mid x > 0\}$,
Range of f^{-1} is $\{y \mid y \in \mathbb{R}\}$

REVIEW SET 3A

1 **a** -1 **b** 27 **c** $\frac{2}{3}$ **2** **a** a^6b^7 **b** $\frac{2}{3x}$ **c** $\frac{y^2}{5}$

3 **a** **i** 81 **ii** $\frac{1}{3}$ **b** $k = 9$

4 **a** $\frac{1}{x^5}$ **b** $\frac{2}{a^2b^2}$ **c** $\frac{2a}{b^2}$ **5** **a** 3^{3-2a} **b** $3^{\frac{5}{2}-\frac{9}{2}x}$

6 **a** 4 **b** $\frac{1}{9}$ **7** **a** $\frac{m}{n^2}$ **b** $\frac{1}{m^3n^3}$ **c** $\frac{m^2p^2}{n}$ **d** $\frac{16n^2}{m^2}$

8 **a** $9 - 6e^x + e^{2x}$ **b** $x - 4$ **c** $2^x + 1$

9 **a** $x = -2$ **b** $x = \frac{3}{4}$ **c** $x = -\frac{1}{4}$

10 **a** C **b** E **c** A **d** B **e** D

11 **a** y^2 **b** y^{-1} **c** $\frac{1}{\sqrt{y}}$ (or $y^{-\frac{1}{2}}$)

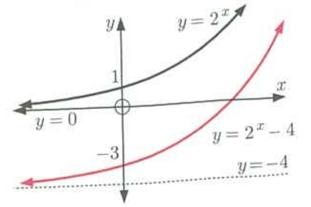
REVIEW SET 3B

1 **a** 2^{n+2} **b** $-\frac{6}{7}$ **c** $3\frac{3}{8}$ **d** $\frac{4}{a^2b^4}$

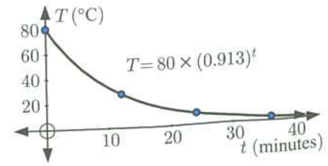
2 **a** 2.28 **b** 0.517 **c** 3.16 **3** **a** 3 **b** 24 **c** $\frac{3}{4}$

4 **a** $\frac{1}{\sqrt{2}} + 1 \approx 1.71$
b $a = -1$

5



6 **a** 80°C
b **i** 26.8°C **ii** 9.00°C **iii** 3.02°C
c



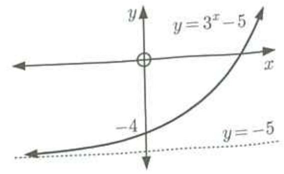
d ≈ 12.8 min

7 **a**

x	-2	-1	0	1	2
y	$-4\frac{8}{9}$	$-4\frac{2}{3}$	-4	-2	4

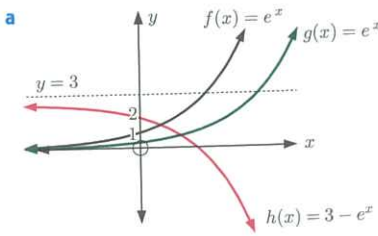
b as $x \to \infty$, $y \to \infty$;
as $x \to -\infty$, $y \to -5$ (above)

c



d $y = -5$

8 **a**



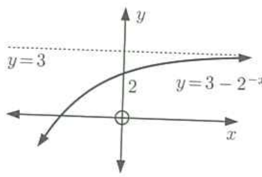
b Domain of f, g, and h is $\{x \mid x \in \mathbb{R}\}$
Range of f is $\{y \mid y > 0\}$, Range of g is $\{y \mid y > 0\}$,
Range of h is $\{y \mid y < 3\}$

9 **a**

x	-2	-1	0	1	2
y	-1	1	2	$2\frac{1}{2}$	$2\frac{3}{4}$

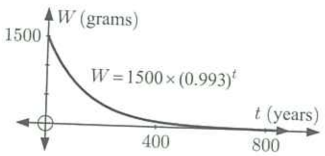
b as $x \to \infty$, $y \to 3$ (below); as $x \to -\infty$, $y \to -\infty$

c



d $y = 3$

10 **a** 1500 g
b **i** 90.3 g **ii** 5.44 g
c



d 386 years

REVIEW SET 3C

1 **a** $x \approx 1.5$ **b** $x \approx -0.6$ **c** $x \approx 1.1$

2 **a** a^{21} **b** p^4q^6 **c** $\frac{4b}{a^3}$

3 **a** 2^{-3} **b** 2^7 **c** 2^{12} **4** **a** $\frac{1}{b^3}$ **b** $\frac{1}{ab}$ **c** $\frac{a}{b}$

5 2^{2x} **6** **a** 5^0 **b** $5^{\frac{3}{2}}$ **c** $5^{-\frac{1}{4}}$ **d** 5^{2a+6}

7 **a** $1+e^{2x}$ **b** $2^{2x}+10(2^x)+25$ **c** $x-49$

8 **a** $x=5$ **b** $x=-4$ **9** $k=\frac{3}{2}$

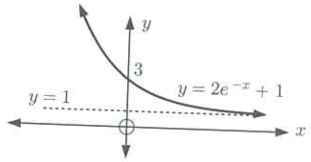
10 **a** $x=4$ **b** $x=-\frac{2}{5}$

11 **a**

x	-2	-1	0	1	2
y	15.8	6.44	3	1.74	1.27

b as $x \to \infty$, $y \to 1$ (above); as $x \to -\infty$, $y \to \infty$

c



d $y = 1$

EXERCISE 4A

1 **a** 4 **b** -3 **c** 1 **d** 0 **e** $\frac{1}{2}$ **f** $\frac{1}{3}$
g $-\frac{1}{4}$ **h** $1\frac{1}{2}$ **i** $\frac{2}{3}$ **j** $1\frac{1}{2}$ **k** $1\frac{1}{3}$ **l** $3\frac{1}{2}$

2 **a** n **b** $a+2$ **c** $1-m$ **d** $a-b$

3 **a** $10^{0.7782}$ **b** $10^{1.7782}$ **c** $10^{3.7782}$ **d** $10^{-0.2218}$
e $10^{-2.2218}$ **f** $10^{1.1761}$ **g** $10^{3.1761}$ **h** $10^{0.1761}$
i $10^{-0.8239}$ **j** $10^{-3.8239}$

4 **a** **i** 0.477 **ii** 2.477 **b** $\log 300 = \log(3 \times 10^2)$

5 **a** **i** 0.699 **ii** -1.301 **b** $\log 0.05 = \log(5 \times 10^{-2})$

6 **a** $x=100$ **b** $x=10$ **c** $x=1$
d $x=\frac{1}{10}$ **e** $x=10^{\frac{1}{2}}$ **f** $x=10^{-\frac{1}{2}}$
g $x=10\,000$ **h** $x=0.000\,01$ **i** $x \approx 6.84$
j $x \approx 140$ **k** $x \approx 0.0419$ **l** $x \approx 0.000\,631$

EXERCISE 4B

1 **a** $10^2=100$ **b** $10^4=10\,000$ **c** $10^{-1}=0.1$
d $10^{\frac{1}{2}}=\sqrt{10}$ **e** $2^3=8$ **f** $3^2=9$
g $2^{-2}=\frac{1}{4}$ **h** $3^{1.5}=\sqrt{27}$ **i** $5^{-\frac{1}{2}}=\frac{1}{\sqrt{5}}$

2 **a** $\log_2 4=2$ **b** $\log_4 64=3$ **c** $\log_5 25=2$
d $\log_7 49=2$ **e** $\log_2 64=6$ **f** $\log_2(\frac{1}{8})=-3$
g $\log_{10} 0.01=-2$ **h** $\log_2(\frac{1}{2})=-1$ **i** $\log_3(\frac{1}{27})=-3$

3 **a** 5 **b** -2 **c** $\frac{1}{2}$ **d** 3 **e** 6 **f** 7 **g** 2
h 3 **i** -3 **j** $\frac{1}{2}$ **k** 2 **l** $\frac{1}{2}$ **m** 5 **n** $\frac{1}{3}$
o n, $a>0$ **p** $\frac{1}{3}$ **q** -1, $t>0$ **r** $\frac{3}{2}$ **s** 0
t 1

4 **a** ≈ 2.18 **b** ≈ 1.40 **c** ≈ 1.87 **d** ≈ -0.0969

5 **a** $x=8$ **b** $x=2$ **c** $x=3$ **d** $x=14$

6 **a** 2 **b** 2 **c** -1 **d** $\frac{3}{4}$ **e** $-\frac{1}{2}$ **f** $\frac{5}{2}$
g $-\frac{3}{2}$ **h** $-\frac{3}{4}$ **i** 2, $x>0$ **j** $\frac{1}{2}$, $x>0$
k 3, $m>0$ **l** $\frac{3}{2}$, $x>0$ **m** -1, $n>0$
n -2, $a>0$ **o** $-\frac{1}{2}$, $a>0$ **p** $\frac{5}{2}$, $m>0$

EXERCISE 4C.1

1 **a** $\log 16$ **b** $\log 20$ **c** $\log 8$ **d** $\log\frac{p}{m}$
e 1 **f** $\log 2$ **g** $\log 24$ **h** $\log_2 6$
i $\log 0.4$ **j** 1 **k** $\log 200$
l $\log(10^t \times w)$ **m** $\log_m\left(\frac{40}{m^2}\right)$ **n** 0
o $\log(0.005)$ **p** $\log_5(\frac{5}{2})$ **q** 2 **r** $\log 28$

2 **a** $\log 96$ **b** $\log 72$ **c** $\log 8$ **d** $\log_3(\frac{25}{8})$
e 1 **f** $\log\frac{1}{2}$ **g** $\log 20$ **h** $\log 25$
i $\log_n\left(\frac{n^2}{10}\right)$

3 **a** 2 **b** $\frac{3}{2}$ **c** 3 **d** $\frac{1}{2}$ **e** -2 **f** $-\frac{3}{2}$

4 For example, for **a**, $\log 9 = \log 3^2 = 2\log 3$

5 **a** $p+q$ **b** $2q+r$ **c** $2p+3q$ **d** $r+\frac{1}{2}q-p$
e $r-5p$ **f** $p-2q$

6 **a** $x+z$ **b** $z+2y$ **c** $x+z-y$ **d** $2x+\frac{1}{2}y$
e $3y-\frac{1}{2}z$ **f** $2z+\frac{1}{4}y-3x$

7 **a** 0.86 **b** 2.15 **c** 1.075

EXERCISE 4C.2

1 **a** $\log y = x\log 2$ **b** $\log y \approx 1.30 + 3\log b$
c $\log M = \log a + 4\log d$ **d** $\log T \approx 0.699 + \frac{1}{2}\log d$
e $\log R = \log b + \frac{1}{2}\log l$ **f** $\log Q = \log a - n\log b$
g $\log y = \log a + x\log b$ **h** $\log F \approx 1.30 - \frac{1}{2}\log n$
i $\log L = \log a + \log b - \log c$ **j** $\log N = \frac{1}{2}\log a - \frac{1}{3}\log b$
k $\log S \approx 2.30 + t\log 2$ **l** $\log y = m\log a - n\log b$

2 **a** $D=2e$ **b** $F=\frac{5}{t}$ **c** $P=\sqrt{x}$ **d** $M=b^2c$
e $B=\frac{m^3}{n^2}$ **f** $N=\frac{1}{\sqrt[3]{p}}$ **g** $P=10x^3$ **h** $Q=\frac{a^2}{x}$

3 **a** $\log_2 y = \log_2 3 + x$ **b** $x = \log_2\left(\frac{y}{3}\right)$
c **i** $x=0$ **ii** $x=2$ **iii** $x \approx 3.32$

4 **a** $x=9$ **b** $x=2$ or 4 **c** $x=25\sqrt{5}$
d $x=200$ **e** $x=5$ **f** $x=3$

EXERCISE 4D.1

1 **a** 2 **b** 3 **c** $\frac{1}{2}$ **d** 0 **e** -1 **f** $\frac{1}{3}$ **g** -2
h $-\frac{1}{2}$

2 **a** 3 **b** 9 **c** $\frac{1}{5}$ **d** $\frac{1}{4}$

3 x does not exist such that $e^x = -2$ or 0

4 **a** a **b** $a+1$ **c** $a+b$ **d** ab **e** $a-b$

5 **a** $e^{1.7918}$ **b** $e^{4.0943}$ **c** $e^{8.6995}$ **d** $e^{-0.5108}$
e $e^{-5.1160}$ **f** $e^{2.7081}$ **g** $e^{7.3132}$ **h** $e^{0.4055}$
i $e^{-1.8971}$ **j** $e^{-8.8049}$

6 **a** $x \approx 20.1$ **b** $x \approx 2.72$ **c** $x=1$
d $x \approx 0.368$ **e** $x \approx 0.006\,74$ **f** $x \approx 2.30$
g $x \approx 8.54$ **h** $x \approx 0.0370$

EXERCISE 4D.2

1 **a** $\ln 45$ **b** $\ln 5$ **c** $\ln 4$ **d** $\ln 24$
e $\ln 1 = 0$ **f** $\ln 30$ **g** $\ln(4e)$ **h** $\ln\left(\frac{6}{e}\right)$
i $\ln 20$ **j** $\ln(4e^2)$ **k** $\ln\left(\frac{20}{e^2}\right)$ **l** $\ln 1 = 0$

2 **a** $\ln 972$ **b** $\ln 200$ **c** $\ln 1 = 0$ **d** $\ln 16$ **e** $\ln 6$
f $\ln\left(\frac{1}{3}\right)$ **g** $\ln\left(\frac{1}{2}\right)$ **h** $\ln 2$ **i** $\ln 16$

3 For example, for **a**, $\ln 27 = \ln 3^3 = 3\ln 3$

4 **Hint:** In **d**, $\ln\left(\frac{e^2}{8}\right) = \ln e^2 - \ln 2^3$

5 **a** $D=ex$ **b** $F=\frac{e^2}{p}$ **c** $P=\sqrt{x}$
d $M=e^3y^2$ **e** $B=\frac{t^3}{e}$ **f** $N=\frac{1}{\sqrt[3]{g}}$
g $Q \approx 8.66x^3$ **h** $D \approx 0.518n^{0.4}$

EXERCISE 4E

1 **a** $x=\frac{1}{\log 2}$ **b** $x=\frac{\log 20}{\log 3}$ **c** $x=\frac{2}{\log 4}$
d $x=4$ **e** $x=-\frac{1}{\log(\frac{3}{4})}$ **f** $x=-5$

2 **a** $x=\ln 10$ **b** $x=\ln 1000$ **c** $x=\ln 0.15$
d $x=2\ln 5$ **e** $x=\frac{1}{2}\ln 18$ **f** $x=0$

3 **a** $t=\frac{\log R - \log 200}{0.25\log 2}$ **b** **i** $t \approx 6.34$ **ii** $t \approx 11.3$

4 **a** $x=\frac{\log M - \log 20}{-0.02\log 5}$ **b** **i** $x=-50$ **ii** $x \approx -76.1$

5 **a** $x=-\frac{\log(0.03)}{\log 2}$ **b** $x=\frac{10\log\left(\frac{10}{3}\right)}{\log 5}$
c $x=\frac{-4\log\left(\frac{1}{8}\right)}{\log 3}$ **d** $x=\frac{1}{2}\ln 42$
e $x=-\frac{100}{3}\ln(0.001)$ **f** $x=\frac{10}{3}\ln\left(\frac{27}{41}\right)$

6 **a** $x=\ln 2$ **b** $x=0$ **c** $x=\ln 2$ or $\ln 3$ **d** $x=0$
e $x=\ln 4$ **f** $x=\ln\left(\frac{3+\sqrt{5}}{2}\right)$ or $\ln\left(\frac{3-\sqrt{5}}{2}\right)$

7 **a** $(\ln 3, 3)$ **b** $(\ln 2, 5)$ **c** $(0, 2)$ and $(\ln 5, -2)$

EXERCISE 4F

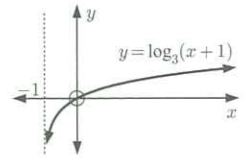
1 **a** ≈ 2.26 **b** ≈ -10.3 **c** ≈ -2.46 **d** ≈ 5.42

2 **a** $x \approx -4.29$ **b** $x \approx 3.87$ **c** $x \approx 0.139$

3 **a** $x=\frac{\log 3}{\log 5}$ **b** $x=\frac{\log\left(\frac{1}{8}\right)}{\log 3}$ **c** $x=-1$ **4** $x=16$

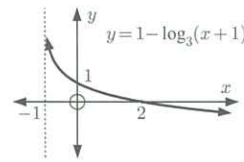
EXERCISE 4G

1 **a** **i** domain is $\{x \mid x > -1\}$, range is $\{y \mid y \in \mathbb{R}\}$
ii VA is $x=-1$, x and y-intercepts 0
iii



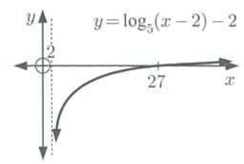
iv $x=-\frac{2}{3}$
v $f^{-1}(x) = 3^x - 1$

b **i** domain is $\{x \mid x > -1\}$, range is $\{y \mid y \in \mathbb{R}\}$
ii VA is $x=-1$, x-intercept 2, y-intercept 1
iii



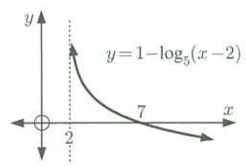
iv $x=8$
v $f^{-1}(x) = 3^{1-x} - 1$

c **i** domain is $\{x \mid x > 2\}$, range is $\{y \mid y \in \mathbb{R}\}$
ii VA is $x=2$, x-intercept 27, no y-intercept
iii



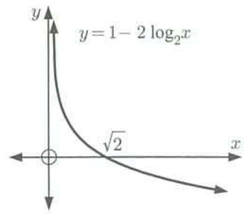
iv $x=7$
v $f^{-1}(x) = 5^{2+x} + 2$

d **i** domain is $\{x \mid x > 2\}$, range is $\{y \mid y \in \mathbb{R}\}$
ii VA is $x=2$, x-intercept 7, no y-intercept
iii



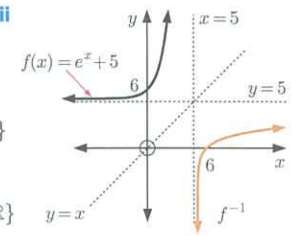
iv $x=27$
v $f^{-1}(x) = 5^{1-x} + 2$

e **i** domain is $\{x \mid x > 0\}$, range is $\{y \mid y \in \mathbb{R}\}$
ii VA is $x=0$, x-intercepts $\pm\sqrt{2}$, no y-intercept
iii



iv $x=\pm 2$
v $f^{-1}(x) = 2^{\frac{1-x}{2}}$

2 **a** **i** $f^{-1}(x) = \ln(x-5)$
ii



iii domain of f is $\{x \mid x \in \mathbb{R}\}$, range is $\{y \mid y > 5\}$, domain of f^{-1} is $\{x \mid x > 5\}$, range is $\{y \mid y \in \mathbb{R}\}$

Mathematical Notation

1. Set Notation

$\in$	is an element of
$\notin$	is not an element of
$\{x_1, x_2, \ldots\}$	the set with elements $x_1, x_2, \ldots$
$\{x : \ldots\}$	the set of all x such that ...
$n(A)$	the number of elements in set A
$\varnothing$	the empty set
$\mathscr{E}$	the universal set
A'	the complement of the set A
$\mathbb{N}$	the set of positive integers, $\{1, 2, 3, \ldots\}$
$\mathbb{Z}$	the set of integers, $\{0, \pm 1, \pm 2, \pm 3, \ldots\}$
$\mathbb{Z}^+$	the set of positive integers, $\{1, 2, 3, \ldots\}$
$\mathbb{R}$	the set of real numbers
$\subseteq$	is a subset of
$\subset$	is a proper subset of
$\nsubseteq$	is not a subset of
$\not\subset$	is not a proper subset of
$\cup$	union
$\cap$	intersection
$[a, b]$	the closed interval $\{x \in \mathbb{R}: a \leqslant x \leqslant b\}$
$[a, b)$	the interval $\{x \in \mathbb{R}: a \leqslant x < b\}$
$(a, b]$	the interval $\{x \in \mathbb{R}: a < x \leqslant b\}$
(a, b)	the open interval $\{x \in \mathbb{R}: a < x < b\}$

2. Miscellaneous Symbols

$=$	is equal to
$\neq$	is not equal to
$\equiv$	is identical to or is congruent to
$\approx$	is approximately equal to
$\propto$	is proportional to
$<$	is less than
$\leq$	is less than or equal to
$\nless$	is not less than
$>$	is greater than
$\geq$	is greater than or equal to
$\ngtr$	is not greater than
∞	infinity

3. Operations

$a + b$	a plus b
$a - b$	a minus b
$a \times b$, ab, $a.b$	a multiplied by b
$a \div b$, $\frac{a}{b}$, a/b	a divided by b
$a : b$	the ratio of a to b
$\sqrt{a}$	the positive square root of the real number a
$\lvert a \rvert$	the modulus of the real number a